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On exceptional quotient singularities

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We study exceptional quotient singularities. In particular, we prove an exceptionality criterion in terms of the α -invariant of Tian, and utilize it to classify four-dimensional and five-dimensional exceptional quotient singularities.

We assume that all varieties are projective, normal, and defined over \mathbb{C} .

1 Introduction

Let X be a smooth Fano variety (see Iskovskikh and Prokhorov [19]) of dimension n , and let $g = g_{i\bar{j}}$ be a Kähler metric with a Kähler form

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum g_{i\bar{j}} dz_i \wedge d\bar{z}_j \in c_1(X).$$

Definition 1.1 The metric g is a Kähler–Einstein metric if $\text{Ric}(\omega) = \omega$, where $\text{Ric}(\omega)$ is a Ricci curvature of the metric g .

Let $\bar{G} \subset \text{Aut}(X)$ be a compact subgroup. Suppose that g is \bar{G} -invariant.

Definition 1.2 Let $P_{\bar{G}}(X, g)$ be the set of C^2 -smooth \bar{G} -invariant functions φ such that

$$\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi > 0$$

and $\sup_X \varphi = 0$. Then the \bar{G} -invariant α -invariant of the variety X is the number

$$\alpha_{\bar{G}}(X) = \sup \left\{ \lambda \in \mathbb{Q} \mid \exists C \in \mathbb{R} \text{ such that } \int_X e^{-\lambda\varphi} \omega^n \leq C \text{ for any } \varphi \in P_{\bar{G}}(X, g) \right\}.$$

The number $\alpha_{\bar{G}}(X)$ was introduced by Tian [42] and Tian and Yau [44] and now it is called the α -invariant of Tian.

Theorem 1.3 [42] *The Fano variety X admits a \bar{G} -invariant Kähler–Einstein metric if $\alpha_{\bar{G}}(X) > n/(n+1)$.*

The normalized Kähler–Ricci flow on the smooth Fano X is defined by the equation

$$(1.4) \quad \begin{cases} \frac{\partial \omega(t)}{\partial t} = -\text{Ric}(\omega(t)) + \omega(t), \\ \omega(0) = \omega, \end{cases}$$

where $\omega(t)$ is a Kähler form such that $\omega(t) \in c_1(X)$, and $t \in \mathbb{R}_{\geq 0}$. It follows from Cao [8] that the solution $\omega(t)$ to (1.4) exists for every $t > 0$.

Theorem 1.5 (Tian–Zhu [45]) *If X admits a Kähler–Einstein metric with a Kähler form ω_{KE} , then any solution to (1.4) converges to ω_{KE} in the sense of Cheeger–Gromov.*

The normalized Kähler–Ricci iteration on the smooth Fano variety X is defined by the equation

$$(1.6) \quad \begin{cases} \omega_{i-1} = \text{Ric}(\omega_i), \\ \omega_0 = \omega, \end{cases}$$

where ω_i is a Kähler form such that $\omega_i \in c_1(X)$. It follows from Yau [46] that the solution ω_i to (1.6) exists for every $i \geq 1$.

Theorem 1.7 (Rubinstein [35]) *If $\alpha_{\bar{G}}(X) > 1$ then X admits a \bar{G} -invariant Kähler–Einstein metric with a Kähler form ω_{KE} and any solution to (1.6) converges to ω_{KE} in $C^\infty(X)$ -topology.*

Smooth Fano varieties that satisfy all hypotheses of Theorem 1.7 do exist.

Example 1.8 Let X be a smooth del Pezzo surface such that $K_X^2 = 5$. Then X is unique and $\text{Aut}(X) \cong S_5$. Moreover, one can show that $\alpha_{\bar{G}}(X) = 2$ in the case when $\bar{G} \cong S_5$ or $\bar{G} \cong A_5$ (see Cheltsov [9, Example 1.11] and Cheltsov and Shramov [11, Theorem A.3]).

Suppose now that $X = \mathbb{P}^n$ (the simplest possible case). Then the Fubini–Study metric on \mathbb{P}^n is Kähler–Einstein. Moreover, if \bar{G} is the maximal compact subgroup of $\text{Aut}(\mathbb{P}^n)$, then the only \bar{G} -invariant metric on \mathbb{P}^n is the Fubini–Study metric and we have $\alpha_{\bar{G}}(\mathbb{P}^n) = +\infty$ by Definition 1.2. In particular, the solution to (1.6) is trivial (and constant) in the latter case, since the initial metric g must be the Fubini–Study metric. On the other hand, the convergence of any solution to (1.6) is not clear in the case when \bar{G} is a finite group. So, Yanir Rubinstein asked the following question in the spring of 2009.

Question 1.9 Is there a finite subgroup $\bar{G} \subset \text{Aut}(\mathbb{P}^n)$ such that $\alpha_{\bar{G}}(\mathbb{P}^n) > 1$?

This paper is inspired by [Question 1.9](#). In particular, we will show that the answer to [Question 1.9](#) is positive in the case when $n \leq 4$, which follows from [\[11, Theorem A.3\]](#) and [Theorems 4.1, 4.2, 4.13, 5.6](#) and [3.21](#).

It came as a surprise that [Question 1.9](#) is strongly related to the notion of exceptional singularity that was introduced by Vyacheslav Shokurov in [\[39\]](#). Let us recall this notion. Let $(V \ni O)$ be a germ of Kawamata log terminal singularity (see [Kollár \[23, Definition 3.5\]](#)).

Definition 1.10 [\[39, Definition 1.5\]](#) The singularity $(V \ni O)$ is said to be *exceptional* if for every effective \mathbb{Q} -divisor D_V on the variety V such that (V, D_V) is log canonical (see [\[23, Definition 3.5\]](#)) and for every resolution of singularities $\pi: U \rightarrow V$ there exists at most one π -exceptional divisor $E \subset U$ such that $a(V, D_V, E) = -1$, where the rational number $a(V, D_V, E)$ can be defined through the equivalence

$$K_U + D_U \sim_{\mathbb{Q}} \pi^*(K_V + D_V) + \sum a(V, D_V, E)E,$$

where the sum is taken over all f -exceptional divisors, and D_U is the proper transform of the divisor D_V on the variety U .

One can show that exceptional Kawamata log terminal singularities are straightforward generalizations of the Du Val singularities of type \mathbb{E}_6 , \mathbb{E}_7 and \mathbb{E}_8 (cf [Theorem 4.1](#)), which partially justifies the word “exceptional” in [Definition 1.10](#).

Remark 1.11 One can easily check (for example, by applying [Theorem 3.11](#)) that the singularity $(V \ni O)$ is not exceptional if V is smooth and $\dim(V) \geq 2$.

It follows from Shokurov [\[38\]](#), Ishii and Prokhorov [\[18\]](#) and Markushevich and Prokhorov [\[27\]](#) that exceptional Kawamata log terminal singularities do exist in dimensions 2 and 3. The existence in dimension 4 follows from Johnson and Kollár [\[20\]](#) and Prokhorov [\[31, Theorem 4.9\]](#). Actually, exceptional Kawamata log terminal singularities exist in every dimension (see [Example 3.13](#)). We will see later (cf [Theorem 1.14](#), [Remark 1.16](#), [Theorem 1.17](#) and [Conjecture 1.23](#)) that [Question 1.9](#) is *almost* equivalent to the following

Question 1.12 Are there exceptional *quotient* singularities of dimension $n + 1$?

Recall that quotient singularities are always Kawamata log terminal by [23, Proposition 3.16]. So Question 1.12 fits well to Definition 1.10. Moreover, it follows from Shokurov [39] and Markushevich and Prokhorov [27] that the answer to Question 1.12 is positive for $n = 1$ and $n = 2$, respectively. The purpose of this paper is to study exceptional *quotient* singularities and, in particular, to give positive answers to Questions 1.9 and 1.12 for every $n \leq 4$. In a subsequent paper we will show that the answers to Questions 1.9 and 1.12 are still positive for $n = 5$ and are surprisingly negative for $n = 6$ (see [10]). So it is hard to predict what would be the answer to Question 1.9 in general. However, we still believe in the following:

Conjecture 1.13 *For every $N \in \mathbb{Z}_{>0}$ there exist exceptional quotient singularities of dimension greater than N .*

Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$, where $n \geq 1$. Denote by $Z(G)$ the center and by $[G, G]$ the commutator of group G . Let $\phi: \mathrm{GL}_{n+1}(\mathbb{C}) \rightarrow \mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}_{n+1}(\mathbb{C})$ be the natural projection. Put $\bar{G} = \phi(G)$ and put

$$\mathrm{lct}(\mathbb{P}^n, \bar{G}) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } (\mathbb{P}^n, \lambda D) \text{ has log canonical singularities} \\ \text{for every } \bar{G}\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n} \end{array} \right. \right\}.$$

Theorem 1.14 (See eg [11, Theorem A.3].) *One has $\mathrm{lct}(\mathbb{P}^n, \bar{G}) = \alpha_{\bar{G}}(\mathbb{P}^n)$.*

The number $\mathrm{lct}(\mathbb{P}^n, \bar{G})$ is usually called \bar{G} -equivariant *global log canonical threshold* of \mathbb{P}^n . Despite the fact that $\mathrm{lct}(\mathbb{P}^n, \bar{G}) = \alpha_{\bar{G}}(\mathbb{P}^n)$, we still prefer to work with the number $\mathrm{lct}(\mathbb{P}^n, \bar{G})$ throughout this paper, because it is easier to handle than $\alpha_{\bar{G}}(\mathbb{P}^n)$. For example, it follows immediately from Definition 3.1 that $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \leq d/(n+1)$ if the group G has a semi-invariant of degree d (a semi-invariant of the group G is a polynomial whose zeroes define a \bar{G} -invariant hypersurface in \mathbb{P}^n).

Remark 1.15 A semi-invariant of the group G is its invariant if $Z(G) \subseteq [G, G]$ and \bar{G} is a nonabelian simple group.

Recall that an element $g \in G$ is called a *reflection* (or sometimes a *quasireflection*) if there is a hyperplane in \mathbb{P}^n that is pointwise fixed by $\phi(g)$ (cf Springer [40, Section 4.1]).

Remark 1.16 Let $R \subseteq G$ be a subgroup generated by all reflections. Then the quotient \mathbb{C}^{n+1}/R is isomorphic to \mathbb{C}^{n+1} (see Shephard and Todd [37] and Springer [40, Theorem 4.2.5]). Moreover, the subgroup $R \subseteq G$ is normal, and the singularity \mathbb{C}^{n+1}/G is isomorphic to the singularity $\mathbb{C}^{n+1}/(G/R)$. Note that the subgroup R is trivial if $G \subset \mathrm{SL}_{n+1}(\mathbb{C})$. If G is a trivial group, then the singularity $\mathbb{C}^{n+1}/G \cong \mathbb{C}^{n+1}$ is not exceptional by Remark 1.11.

Thus to answer [Question 1.12](#) one can always assume that the group G does not contain reflections. On the other hand, one can easily check that there exists a finite subgroup $G' \subset \mathrm{SL}_{n+1}(\mathbb{C})$ such that $\phi(G') = \bar{G}$. So to answer [Question 1.9](#) one can also assume that $G \subset \mathrm{SL}_{n+1}(\mathbb{C})$, which implies, in particular, that the group G does not contain reflections. Moreover, if the group G does not contain reflections, then the singularity \mathbb{C}^{n+1}/G is exceptional if and only if the singularity \mathbb{C}^{n+1}/G' is exceptional thanks to the following:

Theorem 1.17 *Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections. Then*

- *the singularity \mathbb{C}^{n+1}/G is exceptional if $\mathrm{lct}(\mathbb{P}^n, \bar{G}) > 1$,*
- *the singularity \mathbb{C}^{n+1}/G is not exceptional if either $\mathrm{lct}(\mathbb{P}^n, \bar{G}) < 1$ or G has a semi-invariant of degree at most $n + 1$,*
- *for any subgroup $G' \subset \mathrm{GL}_{n+1}(\mathbb{C})$ such that G' does not contain reflections and $\phi(G') = \bar{G}$, the singularity \mathbb{C}^{n+1}/G is exceptional if and only if the singularity \mathbb{C}^{n+1}/G' is exceptional.*

Proof All required assertions immediately follow from [Theorem 3.17](#) (cf [[32](#), Proposition 3.1; [32](#), Lemma 3.1]). \square

It should be pointed out that the assumption that G contains no reflections is crucial for [Theorem 1.17](#).

Example 1.18 Let G be a finite subgroup in $\mathrm{GL}_4(\mathbb{C})$ that is the subgroup number 32 in Shephard and Todd [[37](#), Table VII]. Then the group G is generated by reflections (see [[37](#)]), so that the singularity \mathbb{C}^4/G is not exceptional by [Remark 1.16](#). On the other hand, it follows from [Theorem 4.13](#) that $\mathrm{lct}(\mathbb{P}^3, \bar{G}) \geq 5/4$, because $\bar{G} \cong \mathrm{PSP}_4(\mathbb{F}_3)$. It follows from [Theorem 4.13](#) that there exists a subgroup $G' \subset \mathrm{SL}_4(\mathbb{C})$ such that $\bar{G} = \phi(G')$ and the singularity \mathbb{C}^4/G' is exceptional. One can produce similar examples for two-dimensional and three-dimensional singularities.

By [Theorem 1.17](#) and [[40](#), Section 4.5], if G is a finite subgroup in $\mathrm{GL}_2(\mathbb{C})$ that does not contain reflections, then the singularity \mathbb{C}^2/G is exceptional if and only if G has no semi-invariants of degree at most 2. A similar result holds in dimension 3.

Theorem 1.19 [[27](#), Theorem 1.2] *Let G be a finite group in $\mathrm{GL}_3(\mathbb{C})$ that does not contain reflections. Then the singularity \mathbb{C}^3/G is exceptional if and only if G does not have semi-invariants of degree at most 3.*

For finite subgroups in $\mathrm{GL}_4(\mathbb{C})$, the assertion of [Theorem 1.19](#) is no longer true.

Example 1.20 [[32](#), Example 3.1] Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a binary icosahedron group. Put

$$G = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix} \mid \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \Gamma \ni \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\} \subset \mathrm{SL}_4(\mathbb{C}),$$

where $a_{ij} \in \mathbb{C} \ni b_{ij}$. Then G does not have semi-invariants of degree at most 4, because Γ does not have semi-invariants of degree at most 4 (see [[40](#), Section 4.5]). On the other hand, it follows from [[32](#), Proposition 2.1] that the singularity \mathbb{C}^4/G is not exceptional (cf [Corollary 3.20](#)).

Actually, it is possible to modify the assertion of [Theorem 1.19](#) so that its new version can be generalized to higher dimensions.

Definition 1.21 (Blichfeldt [[3](#)]) The subgroup $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ is said to be primitive if there is no nontrivial decomposition $\mathbb{C}^{n+1} = \bigoplus_{i=1}^r V_i$ such that for any $g \in G$ and any i there is some $j = j(g)$ such that $g(V_i) = V_j$.

If G is primitive, then $\bar{G} \cong G/Z(G)$ by Schur's lemma. It follows from [[32](#), Proposition 2.1] that G must be primitive if \mathbb{C}^{n+1}/G is exceptional (we give a short proof of this fact in [Corollary 3.20](#)). Moreover, primitivity plays a crucial role in the main result of this paper:

Theorem 1.22 Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections. Suppose that $n \leq 4$. Then the following conditions are equivalent:

- The singularity \mathbb{C}^{n+1}/G is exceptional.
- $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \geq (n+2)/(n+1)$.
- The group G is primitive and has no semi-invariants of degree at most $n+1$.

Proof The required assertion follows from Theorems [1.19](#), [3.17](#), [3.18](#), [3.21](#), [4.13](#) and [5.6](#). \square

It appears that in higher dimensions exceptionality cannot be expressed in terms of primitivity and absence of semi-invariants of small degree. In particular, there are nonexceptional six-dimensional quotient singularities arising from primitive subgroups without reflections in $\mathrm{GL}_6(\mathbb{C})$ that have no semi-invariants of degree at most 6 (see

Example 3.25). On the other hand, it follows from [Theorem 1.22](#) that we may expect the sufficient condition for exceptionality in [Theorem 1.17](#) to be a necessary condition as well. Namely, inspired by [Theorem 1.22](#) and Tian [43, Question 1] we believe in the following:

Conjecture 1.23 *Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ that does not contain reflections. Then the singularity \mathbb{C}^{n+1}/G is exceptional if and only if $\mathrm{lct}(\mathbb{P}^n, \bar{G}) > 1$.*

It follows from [Theorem 1.22](#) that [Conjecture 1.23](#) holds for $n \leq 4$. In a subsequent paper we will show that [Conjecture 1.23](#) holds for $n = 5$ and $n = 6$ (see [10]). Note that [Conjecture 1.23](#) is a special case of [Conjecture 3.5](#).

To apply [Theorem 1.22](#) we may assume that $G \subset \mathrm{SL}_{n+1}(\mathbb{C})$, since there exists a finite subgroup $G' \subset \mathrm{SL}_{n+1}(\mathbb{C})$ such that $\phi(G') = \bar{G}$. On the other hand, it is well known that there are at most finitely many primitive finite subgroups in $\mathrm{SL}_{n+1}(\mathbb{C})$ up to conjugation (see Collins [12]). Primitive finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are group-theoretic counterparts of Platonic solids and each of them gives rise to an exceptional singularity (see [Theorem 4.1](#)). Primitive finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ are classified by Blichfeldt in [3]. Prokhorov and Markushevich used Blichfeldt's classification in [27] to obtain an explicit classification of the subgroups in $\mathrm{SL}_3(\mathbb{C})$ corresponding to three-dimensional exceptional quotient singularities (see [Theorem 4.2](#)). For dimension 2 the same was done by Shokurov (see [Theorem 4.1](#)). Similar classification is possible in dimensions 4 and 5, since primitive finite subgroups of $\mathrm{SL}_4(\mathbb{C})$ and $\mathrm{SL}_5(\mathbb{C})$ are classified by Blichfeldt [3] and Brauer [5], respectively. In fact, we obtain a complete list of finite subgroups in $\mathrm{SL}_4(\mathbb{C})$ and $\mathrm{SL}_5(\mathbb{C})$ that satisfy all hypotheses of [Theorem 1.22](#) (see [Theorems 4.13](#) and [5.6](#)).

While the exceptionality of a quotient singularity \mathbb{C}^{n+1}/G depends on a lower bound for a global log canonical threshold $\mathrm{lct}(\mathbb{P}^n, \bar{G})$, it is interesting to find upper bounds for $\mathrm{lct}(\mathbb{P}^n, \bar{G})$ as well. Using [40, Section 4.5; 47] and a bit of direct computation, we see that it follows from [Corollary 3.19](#) that

$$\mathrm{lct}(\mathbb{P}^n, \bar{G}) \leq \begin{cases} 6 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 3 & \text{if } n = 3. \end{cases}$$

Theorem 1.24 *The inequality $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \leq 4(n+1)$ holds for every $n \geq 1$. Moreover, if $n \geq 23$, then $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \leq 12(n+1)/5$.*

Proof Let p be any prime number which does not divide $|G|$. Then G has a semi-invariant of degree at most $(p-1)(n+1)$ by [41, Lemma 2]. Thus, it follows

from Definition 3.1 that $\text{lct}(\mathbb{P}^n, \bar{G}) \leq p - 1$. On the other hand, it follows from the Bertrand's postulate (see Ramanujan [34]) that there is a prime number p' such that $2n + 3 < p' < 2(2n + 3)$, which implies that $p' \leq 4n + 5$. If G is not primitive, then $\text{lct}(\mathbb{P}^n, \bar{G}) \leq 1$ by Corollary 3.19. If G is primitive, then p' does not divide $|G|$ by Feit and Thompson [15, Theorem 1], which completes the proof of the first assertion of the theorem. A similar argument with an additional use of Nagura [29] gives the second assertion for $n \geq 23$. \square

In fact, we expect the following to be true (cf [41]).

Conjecture 1.25 *There exists a universal constant $C \in \mathbb{R}$ such that $\text{lct}(\mathbb{P}^n, \bar{G}) \leq C$ for any finite subgroup $\bar{G} \subset \text{Aut}(\mathbb{P}^n)$ and for any $n \geq 1$.*

Let us describe the structure of the paper. In Section 2 we collect auxiliary results. In Section 3 we prove the exceptionality criterion for a singularity \mathbb{C}^{n+1}/G . In Section 4 we classify exceptional quotient singularities in dimension 4 (see Theorem 4.13). In Section 5 we classify exceptional quotient singularities in dimension 5 (see Theorem 5.6). In Appendix A we prove Corollary A.2 and Theorem A.9 that are used in Section 5.

Many of our results can be obtained by direct computations using the *Atlas of finite groups* [13].

Throughout the paper we use the following standard notation: the symbol \mathbb{Z}_n denotes the cyclic group of order n , the symbol \mathbb{F}_n denotes the finite field consisting of n elements, the symbol S_n denotes the symmetric group of degree n , the symbol A_n denotes the alternating group of degree n , the symbols GL , PGL , SL , PSL , $\text{Sp}_4(\mathbb{F}_3)$ and $\text{PSP}_4(\mathbb{F}_3)$ denote the corresponding algebraic groups. The symbol $k.G$ denotes a central extension of a group G with the center \mathbb{Z}_k (this might be nonunique).

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2 Preliminaries

Throughout the paper we use the standard language of the singularities of pairs (see Kollár [23]). By strictly log canonical singularities we mean log canonical singularities that are not Kawamata log terminal (see [23, Definition 3.5]).

Let X be a variety, let B_X and D_X be effective \mathbb{Q} -divisors on the variety X such that the singularities of the log pair (X, B_X) are Kawamata log terminal, and $K_X + B_X + D_X$ is a \mathbb{Q} -Cartier divisor. Let $Z \subseteq X$ be a closed nonempty subvariety.

Definition 2.1 The log canonical threshold of the boundary D_X along Z is

$$c_Z(X, B_X, D_X) = \sup\{\lambda \in \mathbb{Q} \mid \text{the pair } (X, B_X + \lambda D_X) \text{ is log canonical along } Z\}.$$

Note that the log pair $(X, B_X + D_X)$ is Kawamata log terminal along Z if and only if $c_Z(X, B_X, D_X) > 1$. For simplicity, we put $c(X, B_X, D_X) = c_X(X, B_X, D_X)$. We put $c_Z(X, D_X) = c_Z(X, B_X, D_X)$ in the case when $B_X = 0$. For simplicity, we also put $c(X, D_X) = c_X(X, D_X)$.

Apart from some rare but important occasions (cf Section 3), we only need to consider the case when $B_X = 0$. So from now on we assume that $B_X = 0$.

Let $\pi: \bar{X} \rightarrow X$ be a birational morphism such that \bar{X} is smooth. Then

$$K_{\bar{X}} + D_{\bar{X}} \sim_{\mathbb{Q}} \pi^*(K_X + D_X) + \sum_{i=1}^m d_i E_i,$$

where $D_{\bar{X}}$ is a proper transform of the divisor D_X on the variety \bar{X} , $d_i \in \mathbb{Q}$, and E_i is an exceptional divisor of the morphism π . Put $D_{\bar{X}} = \sum_{i=1}^r a_i \bar{D}_i$, where $a_i \in \mathbb{Q}_{\geq 0}$, and \bar{D}_i is a prime Weil divisor on \bar{X} . Suppose that $\sum_{i=1}^r \bar{D}_i + \sum_{i=1}^m E_i$ is a divisor with simple normal crossing. Put

$$\mathcal{I}(X, D_X) = \pi_* \mathcal{O}_{\bar{X}} \left(\sum_{i=1}^m \lceil d_i \rceil E_i - \sum_{i=1}^r \lfloor a_i \rfloor \bar{D}_i \right),$$

and let $\mathcal{L}(X, D_X)$ be a subscheme that corresponds to the ideal sheaf $\mathcal{I}(X, D_X)$ (the sheaf $\mathcal{I}(X, D_X)$ is an ideal sheaf, because D_X is an effective divisor). Put $\text{LCS}(X, D_X) = \text{Supp}(\mathcal{L}(X, D_X))$.

Remark 2.2 If (X, D_X) is log canonical, then $\mathcal{L}(X, D_X)$ is reduced.

The subscheme $\mathcal{L}(X, D_X)$ and locus $\text{LCS}(X, D_X)$ were introduced by Shokurov [38]. They are called the subscheme of log canonical singularities of the log pair (X, D_X) and the locus of log canonical singularities of the log pair (X, D_X) , respectively. Note that the ideal sheaf $\mathcal{I}(X, D_X)$ is also known as the multiplier ideal sheaf of the log pair (X, D_X) (see Lazarsfeld [25]).

Theorem 2.3 [25, Theorem 9.4.8] *Let H be a nef and big \mathbb{Q} -divisor on X such that $K_X + D_X + H \equiv D$ for some Cartier divisor D on the variety X . Then*

$$H^i(\mathcal{I}(X, D_X) \otimes D) = 0$$

for every $i \geq 1$.

Corollary 2.4 [38, Lemma 5.7] *Suppose that $-(K_X + D_X)$ is nef and big. Then the locus $\text{LCS}(X, D_X)$ is connected.*

Let $\mathbb{LCS}(X, D_X)$ be the set that consists of all possible centers of log canonical singularities of the log pair (X, D_X) (see [11, Definition 2.2]).

Remark 2.5 Let \mathcal{H} be a linear system on the variety X that has no base points. Put $Z \cap H = \sum_{i=1}^k Z_i$, where H is a general divisor in \mathcal{H} , and Z_i is an irreducible subvariety in H . Then $Z \in \mathbb{LCS}(X, D_X)$ if and only if all subvarieties Z_1, \dots, Z_k are contained in the set $\mathbb{LCS}(H, D_X|_H)$.

If $Z \in \mathbb{LCS}(X, D_X)$ and no proper subvariety of Z is contained in $\mathbb{LCS}(X, D_X)$, then Z is said to be a *minimal* center in $\mathbb{LCS}(X, D_X)$ or *minimal* center of log canonical singularities of the log pair (X, D_X) .

Lemma 2.6 (Kawamata [21, Proposition 1.5]) *Suppose that $Z \in \mathbb{LCS}(X, D_X)$ and (X, D_X) is log canonical. Let Z' be a center in $\mathbb{LCS}(X, D_X)$ such that $\emptyset \neq Z \cap Z' = \sum_{i=1}^k Z_i$, where $Z_i \subsetneq Z$ is an irreducible subvariety. Then $Z_i \in \mathbb{LCS}(X, D_X)$ for every $i \in \{1, \dots, k\}$.*

Theorem 2.7 [22, Theorem 1] *Suppose $Z \subset X$ is a minimal center in $\mathbb{LCS}(X, D_X)$ and (X, D_X) is log canonical. Then Z is normal and has at most rational singularities. Let Δ be an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . Then there exists an effective \mathbb{Q} -divisor B_Z on the variety Z such that*

$$(K_X + D_X + \Delta)|_Z \sim_{\mathbb{Q}} K_Z + B_Z,$$

and (Z, B_Z) has Kawamata log terminal singularities.

Let $\bar{G} \subseteq \text{Aut}(X)$ be a finite subgroup such that D_X is \bar{G} -invariant. Then $g(Z) \in \mathbb{LCS}(X, D_X)$ for every $g \in \bar{G}$, and the locus $\mathbb{LCS}(X, D_X)$ is \bar{G} -invariant.

If Z is a minimal center in $\mathbb{LCS}(X, D_X)$ and (X, D_X) is log canonical, then it follows from Lemma 2.6 that

$$Z \cap g(Z) \neq \emptyset \iff Z = g(Z)$$

for every $g \in \bar{G}$.

Lemma 2.8 *Suppose that Z is a minimal center in $\mathbb{LCS}(X, D_X)$, the log pair (X, D_X) is log canonical, and D_X is ample. Let ϵ be an arbitrary rational number such that $\epsilon > 1$. Then there exists an effective \bar{G} -invariant \mathbb{Q} -divisor D on the variety X such that*

$$\mathbb{LCS}(X, D) = \bigcup_{g \in \bar{G}} \{g(Z)\},$$

the log pair (X, D) is log canonical, and the equivalence $D \sim_{\mathbb{Q}} \epsilon D_X$ holds.

Proof Take $m \in \mathbb{Z}$ such that mD_X is a very ample Cartier divisor. Take a general divisor R in the linear system $|nmD_X|$ such that $Z \subset \text{Supp}(R)$ and R is \bar{G} -invariant, where $n \gg 0$. Then

$$\bigcup_{g \in \bar{G}} \{g(Z)\} \subseteq \mathbb{LCS}(X, \lambda D_X + \mu R) \subseteq \mathbb{LCS}(X, D_X)$$

for some positive rational numbers λ and μ such that $\lambda < 1 \leq \lambda + \mu nm < \epsilon$. One has $\lambda D_X + \mu R \sim_{\mathbb{Q}} (\lambda + \mu nm) D_X$.

It follows from the generality of the divisor R that $(X, \mu R)$ is Kawamata log terminal, and

$$\mathbb{LCS}(X, \lambda D_X + \mu R) = \bigcup_{g \in \bar{G}} g(Z),$$

because $\lambda < 1$ and $n \gg 0$. Then there is $\theta \in \mathbb{Q}_{>0}$ such that $0 < 1 - \theta\mu \leq \lambda < 1$ and

$$\bigcup_{g \in \bar{G}} \{g(Z)\} \subseteq \mathbb{LCS}(X, (1 - \theta\mu) D_X + \mu R) \subseteq \mathbb{LCS}(X, \lambda D_X + \mu R),$$

but the log pair $(X, (1 - \theta\mu) D_X + \mu R)$ is log canonical at the general point of Z .

Note that for a fixed R , the number θ is a function of μ . In the above process, we can choose the number μ so that $1 \leq 1 - \theta\mu + \mu nm < \epsilon$ and

$$\mathbb{LCS}(X, (1 - \theta\mu) D_X + \mu R) = \bigcup_{g \in \bar{G}} \{g(Z)\},$$

because Z is a minimal center in $\mathbb{LCS}(X, D_X)$ (see [Lemma 2.6](#)). Put

$$D = (1 - \theta\mu)D_X + \mu R + \frac{\epsilon - 1 - \theta\mu + \mu nm}{nm}M,$$

where M is a general \bar{G} -invariant divisor in $|R|$. Then D is the required divisor. \square

Suppose now that $X = \mathbb{P}^n$. In this case we can say much more about the locus $\mathbb{LCS}(X, D_{\mathbb{P}^n})$ and the set $\mathbb{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})$.

Lemma 2.9 *Let H be a hyperplane in \mathbb{P}^n , and let μ be a nonnegative rational number such that $D_{\mathbb{P}^n} \sim_{\mathbb{Q}} \mu H$. Suppose that the locus $\mathbb{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})$ is an equidimensional subvariety in \mathbb{P}^n of codimension s . Put*

$$r = \begin{cases} \lceil \mu - s - 1 \rceil & \text{if } \mu \notin \mathbb{Z}, \\ \lceil \mu - s - 1 \rceil + 1 & \text{if } \mu \in \mathbb{Z}. \end{cases}$$

Then $r \geq 0$ and

$$\deg(\mathbb{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})) \leq \binom{s+r}{r}.$$

Proof Put $Y = \mathbb{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})$. Let $\Pi \subset \mathbb{P}^n$ be a general linear subspace of dimension s . Put $D = D_{\mathbb{P}^n}|_{\Pi}$ and $\Lambda = H \cap \Pi$. Then $\deg(Y) = |Y \cap \Pi|$ and $\mathbb{LCS}(\Pi, D) = Y \cap \Pi$ by [Remark 2.5](#). One has $K_{\Pi} + D \sim_{\mathbb{Q}} (\mu - s - 1)\Lambda$.

It follows from [Theorem 2.3](#) that there is an exact sequence of cohomology groups

$$0 \longrightarrow H^0(\mathcal{O}_{\Pi}(r\Lambda) \otimes \mathcal{I}(\Pi, D)) \longrightarrow H^0(\mathcal{O}_{\Pi}(r\Lambda)) \longrightarrow H^0(\mathcal{O}_{\mathcal{L}(\Pi, D)}) \longrightarrow 0,$$

and $\text{Supp}(\mathcal{L}(\Pi, D)) = \mathbb{LCS}(\Pi, D) = Y \cap \Pi \neq \emptyset$. Therefore, we see that $r \geq 0$ and

$$\deg(Y) = |Y \cap \Pi| \leq h^0(\mathcal{O}_{\mathcal{L}(\Pi, D)}) \leq h^0(\mathcal{O}_{\Pi}(r\Lambda)) = h^0(\mathcal{O}_{\mathbb{P}^s}(r)) = \binom{s+r}{r},$$

which completes the proof. \square

Let $\phi: \text{GL}_{n+1}(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^n) \cong \text{PGL}_{n+1}(\mathbb{C})$ be the natural projection, and let G be a finite subgroup in $\text{GL}_{n+1}(\mathbb{C})$ such that $\bar{G} = \phi(G)$.

Remark 2.10 If G does not have semi-invariants of degree at most k , then every \bar{G} -orbits in \mathbb{P}^n contains at least $k + 1$ points, because every \bar{G} -orbit consisting of s points defines a \bar{G} -invariant hypersurface in \mathbb{P}^n that is a union of s hyperplanes.

Lemma 2.11 *Let H be a hyperplane in \mathbb{P}^n , and let μ be a nonnegative rational number such that $D_{\mathbb{P}^n} \sim_{\mathbb{Q}} \mu H$. Suppose that G does not have semi-invariants of degree at most $\lfloor \mu \rfloor$. Then $\text{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})$ does not contain subvarieties in \mathbb{P}^n of codimension 1. If in addition $\lfloor \mu \rfloor \leq n+1$ and the log pair $(\mathbb{P}^n, D_{\mathbb{P}^n})$ is log canonical, then $\text{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})$ does not contain points.*

Proof Suppose that $\text{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})$ contains an irreducible subvariety $Y \subset \mathbb{P}^n$ of codimension 1. Let R be the \bar{G} -orbit of the subvariety Y . Then

$$D_{\mathbb{P}^n} = aR + \Delta$$

for some rational number $a \geq 1$ and some effective \mathbb{Q} -divisor Δ on \mathbb{P}^n . Since $D_{\mathbb{P}^n} \sim_{\mathbb{Q}} \mu H$, we see that R is a hypersurface in \mathbb{P}^n of degree at most $\lfloor \mu/a \rfloor \leq \lfloor \mu \rfloor$, which is impossible, because G does not have semi-invariants of degree at most $\lfloor \mu \rfloor$.

We see that $\text{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})$ does not contain subvarieties in \mathbb{P}^n of codimension 1. Let us show that $\text{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})$ does not contain points provided that $\lfloor \mu \rfloor \leq n+1$ and the log pair $(\mathbb{P}^n, D_{\mathbb{P}^n})$ is log canonical.

Suppose that $\lfloor \mu \rfloor \leq n+1$, the log pair $(\mathbb{P}^n, D_{\mathbb{P}^n})$ is log canonical, and $\text{LCS}(\mathbb{P}^n, D_{\mathbb{P}^n})$ contains a point $P \in \mathbb{P}^n$. Let us show that these assumptions lead to a contradiction.

Let Σ be the \bar{G} -orbit of the point P , and let ϵ be a rational number such that $\epsilon > 1$ and $\lfloor \epsilon \mu \rfloor \leq n+1$. Then it follows from [Lemma 2.8](#) that there is an effective \bar{G} -invariant \mathbb{Q} -divisor D on \mathbb{P}^n such that $D \sim_{\mathbb{Q}} \epsilon \mu H$, the log pair (\mathbb{P}^n, D) is log canonical and $\Sigma = \text{LCS}(\mathbb{P}^n, D)$.

Since $\lfloor \epsilon \mu \rfloor \leq n+1$, it follows from [Theorem 2.3](#) that

$$H^0(\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{I}(\mathbb{P}^n, D)) = 0,$$

because $K_{\mathbb{P}^n} + D \sim_{\mathbb{Q}} (\epsilon \mu - n - 1)H$ and $\epsilon \mu - n - 1 < 1$. Therefore, it follows from the exact sequence of cohomology groups

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{I}(\mathbb{P}^n, D)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \longrightarrow H^0(\mathcal{O}_{\Sigma}) \longrightarrow 0$$

that $|\Sigma| \leq n+1$, which is impossible because G does not have semi-invariants of degree at most $\lfloor \mu \rfloor \leq n+1$. \square

Remark 2.12 If G is conjugate to a subgroup in $\text{GL}_{n+1}(\mathbb{R})$, then the subgroup G has an invariant of degree 2, which implies that $\text{lct}(\mathbb{P}^n, \bar{G}) \leq 2/(n+1)$.

Remark 2.13 If Z is a \bar{G} -invariant, then there is a homomorphism $\xi: \bar{G} \rightarrow \text{Aut}(Z)$ that must be a monomorphism provided that Z is not contained in a linear subspace of \mathbb{P}^n , because eigenvectors that correspond to a fixed eigenvalue of any matrix in $\text{GL}_{n+1}(\mathbb{C})$ form a vector subspace in \mathbb{C}^{n+1} .

Theorem 2.14 *Let C be a smooth irreducible curve of genus $g \geq 2$. Then $|\mathrm{Aut}(C)| \leq 84(g-1)$.*

Proof The required inequality is the famous Hurwitz bound (see Breuer [6, Theorem 3.17]). □

3 Exceptionality criterion

Let X be a variety, let B_X be an effective \mathbb{Q} -divisor on X such that the log pair (X, B_X) has at most Kawamata log terminal singularities, and the divisor $-(K_X + B_X)$ is ample. Recall that (X, B_X) is usually called a *log Fano variety*. Let $\bar{G} \subset \mathrm{Aut}(X)$ be a finite subgroup such that the divisor B_X is \bar{G} -invariant.

Definition 3.1 The global \bar{G} -invariant log canonical threshold of the log Fano variety (X, B_X) is a real number $\mathrm{lct}(X, B_X, \bar{G})$ that can be defined as

$$\inf \left\{ c(X, B_X, D_X) \in \mathbb{Q} \;\middle|\; \begin{array}{l} D_X \text{ is a } \bar{G}\text{-invariant } \mathbb{Q}\text{-Cartier effective } \mathbb{Q}\text{-divisor} \\ \text{on the variety } X \text{ such that } D_X \sim_{\mathbb{Q}} -(K_X + B_X) \end{array} \right\}.$$

For simplicity, we put $\mathrm{lct}(X, B_X, \bar{G}) = \mathrm{lct}(X, \bar{G})$ if $B_X = 0$. Similarly, we put $\mathrm{lct}(X, B_X, \bar{G}) = \mathrm{lct}(X, B_X)$ if \bar{G} is trivial. Finally, we put $\mathrm{lct}(X, B_X, \bar{G}) = \mathrm{lct}(X)$ if $B_X = 0$ and \bar{G} is trivial. Then it follows from [11, Theorem A.3] that $\mathrm{lct}(X, \bar{G}) = \alpha_{\bar{G}}(X)$ if X is smooth and $B_X = 0$ (see Definition 1.2).

Remark 3.2 Suppose that $B_X = 0$. Put $V = X/\bar{G}$. Let $\theta: X \rightarrow V$ be the quotient map. Then

$$K_X \sim_{\mathbb{Q}} \theta^*(K_V + R_V),$$

where R_V is a ramification \mathbb{Q} -divisor of the morphism θ . Note that $-(K_V + R_V)$ is an ample \mathbb{Q} -Cartier divisor, and (V, R_V) is Kawamata log terminal by [23, Proposition 3.16]. Moreover, it follows from [23, Proposition 3.16] that $\mathrm{lct}(X, \bar{G}) = \mathrm{lct}(V, R_V)$.

Example 3.3 Suppose that $X \cong \mathbb{P}^1$. Then $B_X = \sum_{i=1}^n a_i P_i$, where P_i is a point, and $a_i \in \mathbb{Q}$ such that $0 \leq a_i < 1$. We may assume that $a_0 \leq \dots \leq a_n$. Then

$$\mathrm{lct}(X, B_X) = \frac{1 - a_n}{2 - \sum_{i=1}^n a_i},$$

where $\sum_{i=1}^n a_i < 2$, because the divisor $-(K_X + B_X)$ is ample. Moreover, it follows from Remark 3.2 that $\mathrm{lct}(X, \bar{G}) = 2/\lambda$, where λ is the length of a \bar{G} -orbit of the smallest length (cf Theorem 4.1).

Lemma 3.4 *The global log canonical threshold $\text{lct}(X, B_X, \bar{G})$ is equal to*

$$\inf \left\{ c \left(X, B_X, \sum_{i=1}^r a_i \mathcal{D}_i \right) \left| \begin{array}{l} \mathcal{D}_i \text{ is a linear system and } a_i \in \mathbb{Q}_{\geq 0} \\ \text{for every } i \in \{1, \dots, r\}, \sum_{i=1}^r a_i \mathcal{D}_i \text{ is } \bar{G}\text{-invariant,} \\ \text{and } \sum_{i=1}^r a_i \mathcal{D}_i \sim_{\mathbb{Q}} -(K_X + B_X) \end{array} \right. \right\}.$$

Proof The required assertion follows from [Definition 3.1](#) and [\[23, Theorem 4.8\]](#). \square

In general, it is unknown whether $\text{lct}(X, B_X, \bar{G})$ is a rational number or not (cf [\[43, Question 1\]](#)). Of course, we expect that $\text{lct}(X, B_X, \bar{G})$ is rational. Moreover, we expect the following to be true.

Conjecture 3.5 *There is an effective \bar{G} -invariant \mathbb{Q} -divisor D_X on X such that $\text{lct}(X, B_X, \bar{G}) = c(X, B_X, D_X) \in \mathbb{Q}$ and $D_X \sim_{\mathbb{Q}} -(K_X + B_X)$.*

Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity, and let $\pi: W \rightarrow V$ be a birational morphism such that the exceptional locus of π consists of one irreducible divisor $E \subset W$ such that $O \in \pi(E)$, the log pair (W, E) has purely log terminal singularities (see [\[23, Definition 3.5\]](#)), and $-E$ is a π -ample \mathbb{Q} -Cartier divisor.

Theorem 3.6 *The birational morphism $\pi: W \rightarrow V$ does exist.*

Proof Modulo the Log Minimal Model Program in dimension $\dim(V)$, the existence of the morphism π follows from [\[31, Proposition 2.9\]](#) in the case when V has \mathbb{Q} -factorial singularities. It follows from [\[24, Theorem 1.5\]](#) that the \mathbb{Q} -factoriality condition in [\[31, Proposition 2.9\]](#) can be removed. Moreover, the proofs of [\[31, Proposition 2.9\]](#) and [\[24, Theorem 1.5\]](#) only need the Log Minimal Model Program for log pairs with big boundaries, which is proved now in [\[2\]](#). \square

We say that $\pi: W \rightarrow V$ is a *plt blow up* of the singularity $(V \ni O)$.

Definition 3.7 [\[31, Definition 4.1\]](#) We say that $(V \ni O)$ is *weakly-exceptional* if it has unique plt blow up.

Weakly-exceptional Kawamata log terminal singularities do exist (see [\[24, Example 2.2\]](#)).

Lemma 3.8 [\[24, Corollary 1.7\]](#) *If $(V \ni O)$ is weakly-exceptional, then $\pi(E) = O$.*

Let R_1, \dots, R_s be irreducible components of $\text{Sing}(W)$ such that $\dim(R_i) = \dim(W) - 2$ and $R_i \subset E$ for every $i \in \{1, \dots, s\}$. Put

$$\text{Diff}_E(0) = \sum_{i=1}^s \frac{m_i - 1}{m_i} R_i,$$

where m_i is the smallest positive integer such that $m_i E$ is Cartier at a general point of R_i .

Lemma 3.9 [23, Theorem 7.5] *The variety E is normal, and $(E, \text{Diff}_E(0))$ is Kawamata log terminal.*

Therefore, if $\pi(E) = O$, then the log pair $(E, \text{Diff}_E(0))$ is a log Fano variety, because $-E$ is π -ample.

Theorem 3.10 [24, Theorem 2.1] *The singularity $(V \ni O)$ is weakly-exceptional if and only if $\pi(E) = O$ and $\text{lct}(E, \text{Diff}_E(0)) \geq 1$.*

Theorem 3.11 [31, Theorem 4.9] *The singularity $(V \ni O)$ is exceptional if and only if $\pi(E) = O$ and $c(E, \text{Diff}_E(0), D_E) > 1$ for every effective \mathbb{Q} -divisor D_E on the variety E such that $D_E \sim_{\mathbb{Q}} -(K_E + \text{Diff}_E(0))$.*

In particular, we see that if the assertion of [Conjecture 3.5](#) is true, then $(V \ni O)$ is exceptional if and only if $\pi(E) = O$ and $\text{lct}(E, \text{Diff}_E(0)) > 1$ holds.

Corollary 3.12 *If $(V \ni O)$ is exceptional, then $(V \ni O)$ is weakly-exceptional.*

It should be pointed out that [Theorem 3.11](#) is an applicable criterion. For instance, it can be used to construct exceptional singularities of any dimension.

Example 3.13 Suppose that $(V \ni O)$ is a Brieskorn–Pham hypersurface singularity

$$\sum_{i=0}^n x_i^{a_i} = 0 \subset \mathbb{C}^{n+1} \cong \text{Spec}(\mathbb{C}[x_0, x_1, \dots, x_n]),$$

where $n \geq 3$ and $2 \leq a_0 < a_1 < \dots < a_n$. Arguing as in the proof of [\[4, Theorem 34\]](#), we see that it follows from [Theorem 3.11](#) that the singularity $(V \ni O)$ is exceptional if

$$1 < \sum_{i=0}^n \frac{1}{a_i} < 1 + \min \left\{ \frac{1}{a_0}, \frac{1}{a_1}, \dots, \frac{1}{a_n} \right\}$$

and a_0, a_1, \dots, a_n are pairwise coprime. This is satisfied if a_0, a_1, \dots, a_n are primes and

$$(3.14) \quad \frac{1}{a_0} + \frac{1}{a_1} + \dots + \frac{1}{a_{n-1}} < 1 < \frac{1}{a_0} + \frac{1}{a_1} + \dots + \frac{1}{a_{n-1}} + \frac{1}{a_n}.$$

We use induction to construct the $(n+1)$ -tuple (a_0, a_1, \dots, a_n) such that a_0, a_1, \dots, a_n are prime integers, and the $(n+1)$ -tuple (a_0, a_1, \dots, a_n) satisfies the inequality (3.14).

If $n = 3$, then the four-tuple $(a_0, a_1, a_2, a_3) = (2, 3, 7, 41)$ satisfies the inequality (3.14).

Suppose that $n \geq 4$, and there are prime numbers $2 \leq c_0 < c_1 < c_2 < \dots < c_{n-1}$ such that

$$\frac{1}{c_0} + \frac{1}{c_1} + \dots + \frac{1}{c_{n-2}} < 1 < \frac{1}{c_0} + \frac{1}{c_1} + \dots + \frac{1}{c_{n-2}} + \frac{1}{c_{n-1}},$$

and assume that $c_{n-1} > 8$ is the largest prime with these properties (for the fixed numbers c_0, \dots, c_{n-2}). It follows from $c_{n-1} > 8$ that there are prime numbers p_1, p_2 and p_3 such that $c_{n-1} < p_1 < p_2 < p_3 < 2c_{n-1}$ (see [34, page 209, (18)]). Put $(a_0, a_1, \dots, a_n) = (c_0, \dots, c_{n-2}, p_2, p_3)$. Then

$$\sum_{i=0}^{n-2} \frac{1}{a_i} + \frac{1}{p_2} < \sum_{i=0}^{n-2} \frac{1}{a_i} + \frac{1}{p_1} \leq 1 < \sum_{i=0}^{n-2} \frac{1}{c_i} + \frac{1}{2c_{n-1}} + \frac{1}{2c_{n-1}} < \sum_{i=0}^{n-2} \frac{1}{a_i} + \frac{1}{p_2} + \frac{1}{p_3}$$

by the maximality assumption imposed on c_{n-1} . So the $(n+1)$ -tuple (a_0, a_1, \dots, a_n) satisfies the inequality (3.14), which completes the construction¹.

Suppose, in addition, that $(V \ni O)$ is a quotient singularity \mathbb{C}^{n+1}/G , where $n \geq 1$ and G is a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$. Put $\bar{G} = \phi(G)$, where $\phi: \mathrm{GL}_{n+1}(\mathbb{C}) \rightarrow \mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}_{n+1}(\mathbb{C})$ is the natural projection.

Remark 3.15 Let $\eta: \mathbb{C}^{n+1} \rightarrow V$ be the quotient map. Then there is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\omega} & W \\ \gamma \downarrow & & \downarrow \pi \\ \mathbb{C}^{n+1} & \xrightarrow{\eta} & V, \end{array}$$

where γ is the blow up of O , the morphism ω is the quotient map that is induced by the lifted action of G on the variety U , and π is a birational morphism. Moreover, π is a plt blow up of the singularity \mathbb{C}^{n+1}/G .

¹ Alternatively, one can use the Sylvester sequence to construct (a_0, \dots, a_n) explicitly (suggested by S. Galkin).

Thus, to prove the existence of a plt blow up of the quotient singularity \mathbb{C}^{n+1}/G we do not need to use [Theorem 3.6](#).

Theorem 3.16 *Suppose that the group $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ does not contain reflections. Then the singularity \mathbb{C}^{n+1}/G is weakly-exceptional if and only if $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \geq 1$.*

Proof Let us use the notation and assumptions of [Remark 3.15](#). Let F be the exceptional divisor of the blow up γ . Put $E = \omega(F)$. Then $F \cong \mathbb{P}^n$ and $E \cong \mathbb{P}^n/\bar{G}$. Since the group G does not contain reflections, it follows from [Remark 3.2](#) that $\mathrm{lct}(\mathbb{P}^n, \bar{G}) = \mathrm{lct}(E, \mathrm{Diff}_E(0))$, which implies that the singularity \mathbb{C}^{n+1}/G is weakly-exceptional if and only if $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \geq 1$ by [Theorem 3.11](#). \square

Theorem 3.17 *Suppose that the group $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ does not contain reflections. Then the singularity \mathbb{C}^{n+1}/G is exceptional if and only if for any \bar{G} -invariant effective \mathbb{Q} -divisor D on \mathbb{P}^n such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$ the log pair (\mathbb{P}^n, D) is Kawamata log terminal.*

Proof Arguing as in the proof of [Theorem 3.16](#) and using [Theorem 3.11](#) together with [\[23, Proposition 3.16\]](#), we obtain the required assertion. \square

Recall that the subgroup $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ is said to be transitive if the corresponding $(n+1)$ -dimensional representation is irreducible (see [\[3\]](#)). Note that G is transitive if it is primitive. As an easy application of [Theorems 3.17](#) and [3.16](#) in conjunction with [Lemma 3.4](#) one can establish the relation between the primitivity of the group G (transitivity, respectively) and the exceptionality of the singularity \mathbb{C}^{n+1}/G (weak-exceptionality, respectively).

Theorem 3.18 *Suppose that the group $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ is not primitive (not transitive, respectively). Then there exists a \bar{G} -invariant effective \mathbb{Q} -divisor D on \mathbb{P}^n such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$ and the pair (\mathbb{P}^n, D) is not Kawamata log terminal (not log canonical, respectively).*

Proof We will only prove that if the group G is not primitive, then there exists a \bar{G} -invariant effective \mathbb{Q} -divisor D on \mathbb{P}^n such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$ and the pair (\mathbb{P}^n, D) is not Kawamata log terminal, since the remaining assertion can be proved similarly.

Suppose that G is not primitive. Then there is a nontrivial decomposition

$$\mathrm{Spec}(\mathbb{C}[x_0, x_1, \dots, x_n]) \cong \mathbb{C}^{n+1} = \bigoplus_{i=1}^r V_i$$

such that $g(V_i) = V_j$ for all $g \in G$. We may assume that $\dim(V_1) \leq \dots \leq \dim(V_r)$. Put $d = \dim(V_1)$. Then $d \leq \lfloor (n+1)/2 \rfloor$. We may assume that $V_1 \subset \mathbb{C}^{n+1}$ is given by $x_d = x_{d+1} = x_{d+2} = \dots = x_n = 0$. Let \mathcal{M}_1 be a linear system on \mathbb{P}^n that consists of hyperplanes that are given by

$$\sum_{i=0}^{d-1} \lambda_i x_i = 0 \subset \mathbb{P}^n \cong \text{Proj}(\mathbb{C}[x_0, x_1, \dots, x_n]),$$

where $\lambda_i \in \mathbb{C}$. Let $\mathcal{M}_1, \dots, \mathcal{M}_s$ be the \bar{G} -orbit of the linear system \mathcal{M}_1 . Then

$$\frac{n+1}{s} \left(\sum_{i=1}^s \mathcal{M}_i \right) \sim_{\mathbb{Q}} -K_{\mathbb{P}^n},$$

where $s \leq \lfloor (n+1)/d \rfloor$. Let $\Lambda \subset \mathbb{P}^n$ be a linear subspace that is given by the equations $x_0 = \dots = x_d = 0$. Then

$$\frac{n+1}{s} \text{mult}_{\Lambda} \left(\sum_{i=1}^s \mathcal{M}_i \right) \geq \frac{n+1}{s} \text{mult}_{\Lambda}(\mathcal{M}_1) = \frac{n+1}{s} \geq d = n - \dim(\Lambda),$$

which implies the desired assertion by [Lemma 3.4](#). \square

Corollary 3.19 *Suppose that the group $G \subset \text{GL}_{n+1}(\mathbb{C})$ is not primitive (not transitive, respectively). Then $\text{lct}(\mathbb{P}^n, \bar{G}) \leq 1$ ($\text{lct}(\mathbb{P}^n, \bar{G}) < 1$, respectively).*

Applying Theorems [3.16](#), [3.17](#) and [3.18](#), we obtain the following.

Corollary 3.20 [[32](#), Proposition 2.1] *Suppose that the group $G \subset \text{GL}_{n+1}(\mathbb{C})$ does not contain reflections. Then the group G is primitive (transitive, respectively) provided that the singularity \mathbb{C}^{n+1}/G is exceptional (weakly-exceptional, respectively).*

Let us show how to apply Theorems [3.16](#) and [3.17](#) (cf [[9](#), Example 1.9]).

Theorem 3.21 *Suppose that $G \subset \text{GL}_3(\mathbb{C})$. Then $\text{lct}(\mathbb{P}^2, \bar{G}) \geq 4/3$ if and only if G does not have semi-invariants of degree at most 3.*

Proof Suppose that the subgroup G does not have semi-invariants of degree at most 3. To complete the proof we must show that $\text{lct}(\mathbb{P}^2, \bar{G}) \geq 4/3$, because the remaining implication is obvious.

Suppose that the strict inequality $\text{lct}(\mathbb{P}^2, \bar{G}) < 4/3$ holds. Then there exist a positive rational number $\lambda < 4/3$ and an effective \bar{G} -invariant \mathbb{Q} -divisor D on \mathbb{P}^2 such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^2}$, and the log pair $(\mathbb{P}^2, \lambda D)$ is strictly log canonical. Applying [Lemma 2.11](#), we obtain a contradiction. \square

Using Theorems 3.17 and 3.21, we obtain the following.

Corollary 3.22 *Suppose that the group $G \subset \mathrm{GL}_3(\mathbb{C})$ does not contain reflections. Then the following are equivalent:*

- *The singularity \mathbb{C}^3/G is exceptional.*
- *The subgroup G does not have semi-invariants of degree at most 3.*
- *The inequality $\mathrm{lct}(\mathbb{P}^2, \bar{G}) \geq 4/3$ holds.*

Arguing as in the proof of Theorem 3.21, we easily obtain a similar assertion that can be used for the classification of three-dimensional weakly exceptional quotient singularities (see [36]).

Theorem 3.23 *Suppose that $G \subset \mathrm{GL}_3(\mathbb{C})$. Then $\mathrm{lct}(\mathbb{P}^2, \bar{G}) \geq 1$ if and only if G does not have semi-invariants of degree at most 2.*

Proof The proof is left to the reader. □

Suppose that $n + 1 = 2l$ for some integer $l \geq 2$. Let $G_1 \subset \mathrm{SL}_2(\mathbb{C})$ and $G_2 \subset \mathrm{SL}_l(\mathbb{C})$ be finite subgroups, let \mathbb{M} be the vector space of $(2 \times l)$ -matrices with entries in \mathbb{C} . For every $(g_1, g_2) \in G_1 \times G_2$ and every $M \in \mathbb{M}$, put

$$(g_1, g_2)(M) = g_1 M g_2^{-1} \in \mathbb{M} \cong \mathbb{C}^{2l},$$

which induces a homomorphism $\varphi: G_1 \times G_2 \rightarrow \mathrm{SL}_{2l}(\mathbb{C})$. Note that $|\ker(\varphi)| \leq 2$ if n is even, and φ is a monomorphism if n is odd.

Lemma 3.24 *Suppose that $G = \varphi(G_1 \times G_2)$. Then $\mathrm{lct}(\mathbb{P}^n, \bar{G}) < 1$.*

Proof Put $s = l - 1$. Let $\psi: \mathbb{P}^1 \times \mathbb{P}^s \rightarrow \mathbb{P}^n$ be the Segre embedding. Put $Y = \psi(\mathbb{P}^1 \times \mathbb{P}^s)$ and let \mathcal{Q} be the linear system consisting of all quadric hypersurfaces in \mathbb{P}^n that pass through the subvariety Y . Then \mathcal{Q} is a nonempty \bar{G} -invariant linear system. The log pair $(\mathbb{P}^n, l\mathcal{Q})$ is not log-canonical along Y , which implies that $\mathrm{lct}(\mathbb{P}^n, \bar{G}) < 1$ by Lemma 3.4. □

As an application of Lemma 3.24 one obtains nonexceptionality of some quotient singularities.

Example 3.25 (cf Theorem 1.22) Suppose that $G = \varphi(G_1 \times G_2)$ and $l = 3$. Then the singularity \mathbb{C}^6/G is not exceptional by Theorem 1.17 and Lemma 3.24. On the other hand, if $G_1 \cong 2.A_5$ and $G_2 \cong 3.A_6$, then G has no semi-invariants of degree at most 6 which can be shown by direct computation.

Suppose that $l = 2$. The transposition of matrices in \mathbb{M} induces an involution $\iota \in \mathrm{SL}_4(\mathbb{C})$.

Lemma 3.26 *If G is generated by $\varphi(G_1 \times G_2)$ and ι , then $\mathrm{lct}(\mathbb{P}^3, \bar{G}) < 1$.*

Proof See the proof of [Lemma 3.24](#). □

4 Four-dimensional case

Shokurov [\[38\]](#) and Prokhorov and Markushevich [\[27\]](#) obtained an explicit classification of exceptional quotient singularities of dimension 2 and 3. Namely, for Gorenstein quotient singularities they prove the following.

Theorem 4.1 [\[38, Example 5.2.3\]](#) *Let G be the finite subgroup in $\mathrm{SL}_2(\mathbb{C})$. Then the singularity \mathbb{C}^2/G is exceptional if and only if G is a binary central extension of one of the following groups: A_4 , S_4 or A_5 .*

Theorem 4.2 [\[27, Theorem 3.13\]](#) *Let G be a finite subgroup in $\mathrm{SL}_3(\mathbb{C})$. Then the singularity \mathbb{C}^3/G is exceptional if and only if G is one of the following subgroups:*

- a central extension of $\mathrm{PSL}_2(\mathbb{F}_7)$, which is isomorphic to either $\mathrm{PSL}_2(\mathbb{F}_7)$ or $\mathbb{Z}_3 \times \mathrm{PSL}_2(\mathbb{F}_7)$,
- a nontrivial central extension $3.A_6$ of the alternating group A_6 by \mathbb{Z}_3 ,
- the Hessian group, which can be characterized by the exact sequence

$$1 \longrightarrow \mathbb{H}(3, \mathbb{F}_3) \longrightarrow G \longrightarrow S_4 \longrightarrow 1,$$

where $\mathbb{H}(3, \mathbb{F}_3)$ is the Heisenberg group consisting of all unipotent (3×3) -matrices with entries in \mathbb{F}_3 ,

- the normal subgroup of the Hessian group of index 3 that contains $\mathbb{H}(3, \mathbb{F}_3)$.

The purpose of this section is to present an analogous classification for exceptional singularities of dimension 4 (see [Theorem 4.13](#)), and prove some relevant results.

Let \bar{G} be a finite subgroup in $\mathrm{Aut}(\mathbb{P}^3)$, and let $\phi: \mathrm{GL}_4(\mathbb{C}) \rightarrow \mathrm{Aut}(\mathbb{P}^3)$ be the natural projection. Then there is a finite subgroup in $\mathrm{SL}_4(\mathbb{C})$ such that $\phi(G) = \bar{G}$. Moreover, if G is primitive, then it follows from [\[3; 14\]](#) that one may assume that $Z(G) \subseteq [G, G]$, where $Z(G)$ and $[G, G]$ are the center and the commutator of the group G , respectively.

As a warming-up we start with a result that can be applied to a classification of four-dimensional weakly exceptional quotient singularities (see [\[36\]](#)).

Theorem 4.3 *The inequality $\text{lct}(\mathbb{P}^3, \bar{G}) \geq 1$ holds if and only if the following three conditions are satisfied: the group G is transitive, the group G does not have semi-invariants of degree at most 3, and² there is no \bar{G} -invariant smooth rational cubic curve in \mathbb{P}^3 .*

Proof Let us prove the \Rightarrow part. If G has a semi-invariant of degree at most 3, then $\text{lct}(\mathbb{P}^3, \bar{G}) \leq 3/4$ by [Definition 3.1](#). If G is not transitive, then $\text{lct}(\mathbb{P}^3, \bar{G}) < 1$ by [Corollary 3.19](#).

Suppose that there is a \bar{G} -invariant smooth rational cubic curve $C \subset \mathbb{P}^3$. Let $R \subset \mathbb{P}^3$ be the surface that is swept out by lines that are tangent to C . Then $c(\mathbb{P}^3, R) = 5/6$ the surface R is \bar{G} -invariant, and $\deg(R) = 4$. Hence, we see that $\text{lct}(\mathbb{P}^3, \bar{G}) \leq 5/6$.

Let us prove the \Leftarrow part. Suppose that G is transitive, the subgroup G has no semi-invariants of degree at most 3, there is no \bar{G} -invariant smooth rational cubic curve in \mathbb{P}^3 , but $\text{lct}(\mathbb{P}^3, \bar{G}) < 1$.

There is an effective \bar{G} -invariant \mathbb{Q} -divisor D on \mathbb{P}^3 such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ and a positive rational number $\lambda < 1$ such that $(\mathbb{P}^3, \lambda D)$ is strictly log canonical. Let S be an irreducible subvariety of \mathbb{P}^3 that is a minimal center in $\text{LCS}(\mathbb{P}^3, \lambda D)$. By [Lemma 2.8](#), we may assume that

$$\text{LCS}(\mathbb{P}^3, \lambda D) = \bigcup_{g \in \bar{G}} \{g(S)\},$$

where $\dim(S) \neq 2$, because G has no semi-invariants of degree at most 3.

The locus $\text{LCS}(\mathbb{P}^3, \lambda D)$ is connected by [Corollary 2.4](#). Then S is \bar{G} -invariant by [Lemma 2.6](#). Since the group G is transitive, we see that S is not a point. We see that S is a curve. Then $\deg(S) \leq 3$ by [Lemma 2.9](#), and S is not contained in a plane, because G is transitive. Hence S is a smooth rational cubic curve. \square

Combining [Remark 2.13](#), [Theorem 4.3](#) and the classification of finite subgroups in $\text{PGL}_2(\mathbb{C})$, we easily obtain the following result (cf [Theorem 3.23](#)).

Corollary 4.4 *The inequality $\text{lct}(\mathbb{P}^3, \bar{G}) \geq 1$ holds if the following three conditions are satisfied: the group G is transitive, the group G does not have semi-invariants of degree at most 3, and the group \bar{G} is not isomorphic to the alternating group A_5 .*

²One can show that the third condition of [Theorem 4.3](#) is not redundant. Namely, if $G \subset \text{SL}_4(\mathbb{C})$ is a primitive group isomorphic to $2.A_5$, then G has no semi-invariants of degree at most 3, but there is a \bar{G} -invariant twisted cubic in \mathbb{P}^3 . In fact, the primitive group $G \cong 2.A_5$ gives essentially the only example of this kind.

The main purpose of this section is to prove the following result (cf [Theorem 1.19](#)).

Theorem 4.5 *The inequality $\text{lct}(\mathbb{P}^3, \bar{G}) \geq 5/4$ holds if the following three conditions are satisfied: the group G is primitive, the group G does not have semi-invariants of degree at most 4, and the inequality $|\bar{G}| \geq 169$ holds.*

Proof Suppose that G is primitive and does not have semi-invariants of degree at most 4, the inequality $|\bar{G}| \geq 169$ holds, but $\text{lct}(\mathbb{P}^3, \bar{G}) < 5/4$. Let us derive a contradiction.

There is an effective \bar{G} -invariant \mathbb{Q} -divisor D on \mathbb{P}^3 such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ and a positive rational number $\lambda < 5/4$ such that $(\mathbb{P}^3, \lambda D)$ is strictly log canonical.

Let S be an irreducible subvariety in \mathbb{P}^3 that is a minimal center in $\text{LCS}(\mathbb{P}^3, \lambda D)$. Then S is a curve by [Lemma 2.11](#).

Note that $g(S) \in \text{LCS}(\mathbb{P}^3, \lambda D)$ for every $g \in \bar{G}$, because the divisor D is \bar{G} -invariant. It follows from [Lemma 2.6](#) that

$$S \cap g(S) \neq \emptyset \iff S = g(S)$$

for every $g \in \bar{G}$. It follows from [Lemma 2.8](#) that we may assume that

$$\text{LCS}(\mathbb{P}^3, \lambda D) = \bigcup_{g \in \bar{G}} \{g(S)\}.$$

Let \mathcal{I} be the multiplier ideal sheaf of the log pair $(\mathbb{P}^3, \lambda D)$, and let \mathcal{L} be the log canonical singularities subscheme of the log pair $(\mathbb{P}^3, \lambda D)$. Then there is an exact sequence

$$(4.6) \quad 0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{I}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow 0$$

by [Theorem 2.3](#). Then it follows from [Theorem 2.7](#) that S is a smooth curve of genus g such that $2g - 2 < \deg(S)$.

Let Z be the \bar{G} -orbit of the curve S . Then Z is smooth and $\deg(Z) \leq 6$ by [Lemma 2.9](#). Then $2g - 2 < \deg(S) \leq 6$, which implies that $g \leq 3$. Note that $Z = \mathcal{L}$ by [Remark 2.2](#), because $(\mathbb{P}^3, \lambda D)$ is log canonical. Moreover, the curve Z is not contained in a plane, because G is transitive.

Let r be the number of irreducible components of Z . Then $6 \geq \deg(Z) = r \deg(S)$, which implies that $r \leq 6$. Note that $r = 0$ if $g \geq 3$.

Using (4.6) and the Riemann–Roch theorem, we see that

$$(4.7) \quad 4 = h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) = r(\deg(S) - g + 1),$$

because $\mathcal{L} = Z$ and $2g - 2 < \deg(S)$. In particular, we see that $r \leq 2$.

One has $\deg(S) \neq 1$, because G is primitive. Thus S is not contained in a plane, because otherwise the \bar{G} -orbit of the plane spanned by S would give a semi-invariant of G of degree 1 or 2. Thus, we have $6 \geq \deg(Z) = r \deg(S) \geq 3r$.

If $r = 2$, then $\deg(S) = 3$ and $g = 0$, which contradicts the equality (4.7). We see that $r = 1$ and $Z = S$. Then $g \leq 1$ by Theorem 2.14 and Remark 2.13, because $|\bar{G}| \geq 169$.

Arguing as in the proof of Theorem 4.3, we see that $g \neq 0$, because G does not have semi-invariants of degree 4. Then it follows from (4.7) that $g = 1$ and $\deg(S) = 4$. We see that $S = Q_1 \cap Q_2$, where Q_1 and Q_2 are irreducible quadrics in \mathbb{P}^3 .

Let \mathcal{P} be a pencil generated by Q_1 and Q_2 . Then \mathcal{P} contains exactly 4 singular surfaces, which are simple quadric cones. This means that there is a \bar{G} -orbit in \mathbb{P}^3 consisting of at most 4 points, which is impossible by Remark 2.10. □

In the rest of this section we will refine the assertion of Theorem 4.5 by removing the assumption that \bar{G} contains at least 169 elements and providing an explicit list of possible finite subgroups in $\mathrm{PGL}_4(\mathbb{C})$ that satisfy all hypothesis of Theorem 4.5 (cf Theorems 4.1 and 4.2). Let us start with the following example.

Example 4.8 (See Blichfeldt [3, Section 123] and Nieto [30].) Let \mathbb{H} be a subgroup in $\mathrm{SL}_4(\mathbb{C})$ that is conjugate to the subgroup generated by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and let $N \subset \mathrm{SL}_4(\mathbb{C})$ be the normalizer of the subgroup \mathbb{H} . There is an exact sequence of groups³

$$1 \longrightarrow \tilde{\mathbb{H}} \xrightarrow{\alpha} N \xrightarrow{\beta} S_6 \longrightarrow 1,$$

where $\tilde{\mathbb{H}} = \langle \mathbb{H}, \mathrm{diag}(\sqrt{-1}) \rangle$. One can show that N is a primitive subgroup of $\mathrm{SL}_4(\mathbb{C})$.

³The choice of the epimorphism β is not canonical even up to conjugation, due to the existence of outer automorphisms of S_6 . There are essentially two possible choices of β . To fix one of them we use the fact that the subspace $W \subset \mathrm{Sym}^4(\mathbb{C}^4)$ of $\tilde{\mathbb{H}}$ -invariant quartics is five-dimensional; moreover, the group $N/\tilde{\mathbb{H}}$ acts on W , and W is an irreducible representation of $N/\tilde{\mathbb{H}}$ (cf the proof of Lemma 4.12 and references therein). We choose β so that W corresponds to the standard five-dimensional representation of S_6 twisted by the sign representation. Another way to describe the choice of β is through introducing the action of $N/\tilde{\mathbb{H}}$ on the space $W' = \Lambda^2(\mathbb{C}^4)$ (see [30]).

The following theorem provides an explicit list of possible finite subgroups in $\mathrm{PGL}_4(\mathbb{C})$ that satisfy all hypotheses of [Theorem 4.5](#):

Theorem 4.9 (See [[3](#), Chapter VII; [14](#), Section 8.5].) *Let G be a primitive subgroup of $\mathrm{SL}_4(\mathbb{C})$ such that $Z(G) \subseteq [G, G]$. Then one of the following possibilities holds:*

- either G satisfies the hypotheses of [Lemma 3.24](#) or [Lemma 3.26](#),
- or G is one of the following groups:
 - A_5 or S_5 ,
 - $\mathrm{SL}_2(\mathbb{F}_5)$,
 - $\mathrm{SL}_2(\mathbb{F}_7)$,
 - $2.A_6$, which is a central extension of the group $A_6 \cong \bar{G}$,
 - $2.S_6$, which is a central extension⁴ of the group $S_6 \cong \bar{G}$,
 - $2.A_7$, which is a central extension of the group $A_7 \cong \bar{G}$,
 - $\mathrm{Sp}_4(\mathbb{F}_3)$,
 - in the notation of [Example 4.8](#), a primitive subgroup in N that contains $\alpha(\tilde{\mathbb{H}})$.

It should be pointed out that [Theorem 4.9](#) describes primitive subgroups of $\mathrm{SL}_4(\mathbb{C})$ up to conjugation. Namely, if there are two monomorphisms $\iota_1: G \rightarrow \mathrm{SL}_4(\mathbb{C})$ and $\iota_2: G \rightarrow \mathrm{SL}_4(\mathbb{C})$ such that both subgroups $\iota_1(G)$ and $\iota_2(G)$ are primitive, then it follows from [[3](#), Chapter VII] that $\iota_1(G)$ and $\iota_2(G)$ are conjugate, but it may happen that the representations of the group G given by ι_1 and ι_2 are nonisomorphic, ie there is no element $g \in \mathrm{SL}_4(\mathbb{C})$ that makes the diagram

$$\begin{array}{ccc} & G & \\ \iota_1 \swarrow & & \searrow \iota_2 \\ \mathrm{SL}_4(\mathbb{C}) & \xrightarrow{\theta_g} & \mathrm{SL}_4(\mathbb{C}) \end{array}$$

commutative, where θ_g is the conjugation by g (cf [[13](#)]).

Lemma 4.10 *Suppose that $G \cong 2.A_6$. Then G has no semi-invariants of degree at most 4.*

Proof Semi-invariants of G are its invariants by [Remark 1.15](#), and G has no odd degree invariants, because G contains a scalar matrix whose nonzero entries are -1 .

To complete the proof, it is enough to prove that G has no invariants of degree 4.

⁴There are three nonisomorphic nontrivial central extensions of the group S_6 with the center isomorphic to \mathbb{Z}_2 , two of which are embedded in $\mathrm{SL}_4(\mathbb{C})$ (cf [[13](#)]). But up to conjugation there is only one subgroup of $\mathrm{PGL}_4(\mathbb{C})$ isomorphic to S_6 .

Let $V \cong \mathbb{C}^4$ be the irreducible representation of the group G that corresponds to the embedding $G \subset \mathrm{SL}_4(\mathbb{C})$. Without loss of generality, we may assume that $\Lambda^2 V \cong \mathbb{C}^6$ is a permutation representation of the group $G/Z(G) \cong A_6$, because G has two four-dimensional irreducible representations, which give one subgroup $G \subset \mathrm{SL}_4(\mathbb{C})$ up to conjugation.

Let χ be the character of the representation V , and let χ_4 be the character of the representation $\mathrm{Sym}^4(V)$. Then

$$\chi_4(g) = \frac{1}{24}(\chi(g)^4 + 6\chi(g)^2\chi(g^2) + 3\chi(g^2)^2 + 8\chi(g)\chi(g^3) + 6\chi(g^4))$$

for every $g \in G$. The values of the characters χ and χ_4 are listed in Table 1. In this

	$[5, 1]_{10}$	$[5, 1]_5$	$[4, 2]_8$	$[3, 3]_6$	$[3, 3]_3$	$[3, 1, 1, 1]_6$	$[3, 1, 1, 1]_3$	$[2, 2, 1, 1]_4$	z	e
#	144	144	180	40	40	40	40	90	1	1
χ	1	−1	0	−1	1	2	−2	0	−4	4
χ_4	0	0	−1	2	2	−4	−4	3	35	35

Table 1

table, the first row lists the types of the elements in G (for example, the symbol $[5, 1]_{10}$ denotes the set⁵ of order 10 elements whose image in A_6 is a product of disjoint cycles of length 5 and 1), and z and e are the nontrivial element in the center of G and the identity element, respectively.

Now one can check that the inner product of the character χ_4 and the trivial character is zero, which implies that the subgroup G does not have invariants of degree 4. \square

Lemma 4.11 *If $G \cong 2.S_6$ or $G \cong 2.A_7$, then G has no semi-invariants of degree at most 4.*

Proof Recall that these groups contain $2.A_6$ and we can apply Lemma 4.10. \square

Lemma 4.12 *Under the assumptions of Theorem 4.9 the subgroup G has no semi-invariants of degree at most 4 if and only if G is one of the following groups:*

- $2.A_6$, $2.S_6$ or $2.A_7$,
- $\mathrm{Sp}_4(\mathbb{F}_3)$,

⁵ Note that these sets do not coincide with conjugacy classes. For example, the image of the set of the elements of type $[5, 1]_{10}$ under the natural projection $2.A_6 \rightarrow A_6$ is a union of two different conjugacy classes in A_6 .

- in the notation of [Example 4.8](#), a subgroup of N that satisfies one of the following four conditions:
 - $G = N$,
 - $\alpha(\tilde{\mathbb{H}}) \subsetneq G$ and $\beta(G) \cong A_6$,
 - $\alpha(\tilde{\mathbb{H}}) \subsetneq G$ and $\beta(G) \cong S_5$, where the embedding $\beta(G) \subset S_6$ is nonstandard, ie the standard one twisted by an outer automorphism of S_6 ,
 - $\alpha(\tilde{\mathbb{H}}) \subsetneq G$ and $\beta(G) \cong A_5$, where the embedding $\beta(G) \subset S_6$ is nonstandard.

Proof Let d be the smallest positive number such G has an semi-invariant of degree d . If $G \cong 2.A_6$, then $d \geq 5$ by [Lemma 4.10](#). If $G \cong 2.S_6$ or $G \cong 2.A_7$, then $d \geq 5$ by [Lemma 4.11](#). In fact, one can check by direct computation that $d = 8$ if $G \cong 2.A_6$ or $G \cong 2.S_6$ or $G \cong 2.A_7$. If $G \cong \mathrm{SL}_2(\mathbb{F}_7)$, then the equality $d = 4$ holds by [\[26\]](#) and [Remark 1.15](#). If $G \cong \mathrm{Sp}_4(\mathbb{F}_3)$, then the equality $d = 12$ holds by [\[28\]](#) and [Remark 1.15](#).

Suppose that $G \cong \mathrm{SL}_2(\mathbb{F}_5) \cong 2.A_5$. Then there is a \bar{G} -invariant smooth rational cubic curve $C \subset \mathbb{P}^3$, because the representation $G \rightarrow \mathrm{GL}_4(\mathbb{C})$ is a symmetric square of a two-dimensional representation of the group G . The surface swept out by the lines tangent to the curve C is a \bar{G} -invariant surface of degree 4 (cf proof of [Theorem 4.3](#)). Therefore, the inequality $d \leq 4$ holds⁶.

Let us use the notation of [Example 4.8](#). By [Theorem 4.9](#), [Remark 2.12](#) and [Lemmas 3.24](#) and [3.26](#), to complete the proof we may assume that G is a primitive subgroup in N that contains $\alpha(\tilde{\mathbb{H}})$.

One can show that the group $\tilde{\mathbb{H}}$ has no invariants of degree less than 4 and its invariants of degree 4 form a five-dimensional vector space W (see eg [\[33, Lemma 3.18\]](#)).

The group $\beta(G)$ naturally acts on W . Moreover, the subgroup G has an invariant of degree 4 if and only if the representation W has a one-dimensional subrepresentation of the group $\beta(G)$. On the other hand, it follows from [\[30\]](#) that if $G = N$, then W is an irreducible representation of $\beta(G) = S_6$.

It follows from [\[3, Section 123\]](#) that, up to conjugation, there exist exactly 9 possibilities for the subgroup $G \subset N$ such that G is primitive. These possibilities are listed in [Table 2](#). In this table, the first column lists the labels of the subgroup G according to [\[3, Section 123\]](#) and the last column lists the dimensions of the irreducible $\beta(G)$ -subrepresentations of W .

Note that $\mathbb{H} \subset \tilde{\mathbb{H}}$ has no semi-invariants of degree 3, because \mathbb{H} has no invariants of degree 3, the center of the group \mathbb{H} coincides with its commutator and acts nontrivially on cubic forms.

⁶Actually, one can show that $d = 4$ in this case.

Label of the group G	$\beta(G)$	Generators of the subgroup $\beta(G) \subseteq S_6$	Splitting type
13°	\mathbb{Z}_5	(24635)	$1, 1, 1, 1, 1$
14°	$\mathbb{Z}_5 \rtimes \mathbb{Z}_2$	$(24635), (36)(45)$	$1, 2, 2$
15°	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$(24635), (3465)$	$1, 2, 2$
16°	A_5	$(24635), (34)(56)$	$1, 4$
17°	A_5	$(24635), (12)(36)$	5
18°	S_5	$(24635), (56)$	$1, 4$
19°	S_5	$(24635), (12)(34)(56)$	5
20°	A_6	$(24635), (12)(34)$	5
21°	S_6	$(24635), (12)$	5

Table 2

The subgroups of N described in Lemma 4.12 are the subgroups 21° , 20° , 19° , 17° , respectively. We see that $d \leq 4$ if G is the subgroup 13° , 14° , 15° , 16° or 18° . On the other hand, if G is the subgroup 17° , 19° , 20° or 21° , then the subgroup G has neither semi-invariants of degree less than 4, nor invariants of degree 4. Let us prove that the subgroup 17° does not have semi-invariants of degree 4. Since the absence of semi-invariants of degree 4 implies the absence of semi-invariants of degree 2, this would imply that in the case when G is the subgroup 17° , 19° , 20° or 21° of the group N the inequality $d \geq 5$ holds⁷.

Suppose that G is the subgroup 17° , and suppose, in addition, that G does have a semi-invariant Φ of degree 4. Let us show that this assumption leads to a contradiction.

Note that the polynomial Φ is not $\tilde{\mathbb{H}}$ -invariant, because Φ is not G -invariant and $G/\tilde{\mathbb{H}} \cong \beta(G) \cong A_5$ is a simple group. Let Z be the center of the group $\tilde{\mathbb{H}}$. Put $\bar{\mathbb{H}} = \phi(\tilde{\mathbb{H}})$. Then $\bar{\mathbb{H}}/Z \cong \bar{\mathbb{H}} \cong \mathbb{Z}_2^4$, and Z acts trivially on Φ . Thus, there is a homomorphism $\xi: \bar{\mathbb{H}} \rightarrow \mathbb{C}^*$ such that $\ker(\xi) \neq \bar{\mathbb{H}}$, which implies that $\ker(\xi) \cong \mathbb{Z}_2^3$, because $\text{im}(\chi)$ is a cyclic group. Let $\theta: \bar{G} \rightarrow \text{Aut}(\bar{\mathbb{H}})$ be the homomorphism such that

$$\theta(g)(h) = ghg^{-1} \in \bar{\mathbb{H}} \cong \mathbb{Z}_2^4$$

for all $g \in \bar{G}$ and $h \in \bar{\mathbb{H}}$. Consider $\bar{\mathbb{H}}$ as a vector space over \mathbb{F}_2 . Then θ induces a monomorphism $\tau: \beta(G) \rightarrow \text{GL}_4(\mathbb{F}_2)$ and $\ker(\xi)$ is a $\text{im}(\tau)$ -invariant subspace. But $\text{im}(\tau) \cong A_5$ has no nontrivial three-dimensional representations over \mathbb{F}_2 , because

⁷In fact, one can check by direct computation that $d = 8$ if G is the subgroup 17° , 19° , 20° or 21° .

$|\mathrm{GL}_3(\mathbb{F}_2)| = 168$ is not divisible by $|A_5| = 60$. Thus, we see that there is a nonzero element $t \in \overline{\mathbb{H}}$ such that t is $\mathrm{im}(\tau)$ -invariant. Let F be the stabilizer of t in $\mathrm{GL}_4(\mathbb{F}_2)$. Then $A_5 \cong \mathrm{im}(\tau) \subset F$, which is impossible, because $|F| = 1344$ is not divisible by $|A_5| = 60$. \square

Combining the previous results we obtain the following.

Theorem 4.13 *Let G be a finite subgroup in $\mathrm{SL}_4(\mathbb{C})$. Then the following conditions are equivalent:*

- *The singularity $(V \ni O)$ is exceptional.*
- *The inequality $\mathrm{lct}(\mathbb{P}^3, \overline{G}) \geq 5/4$ holds.*
- *The group G is primitive and G does not have semi-invariants of degree at most 4.*
- *$\overline{G} = \phi(G')$, where G' is one of the 8 subgroups listed in [Lemma 4.12](#).*

Proof This follows from Theorems [1.17](#), [4.5](#) and [4.9](#) and [Lemma 4.12](#). \square

5 Five-dimensional case

The purpose of this section is to present an explicit classification of exceptional five-dimensional singularities (see [Theorem 5.6](#), cf Theorems [4.1](#), [4.2](#) and [4.13](#)), and prove some relevant results.

Let \overline{G} be a finite subgroup in $\mathrm{Aut}(\mathbb{P}^4)$, and consider the natural projection

$$\phi: \mathrm{SL}_5(\mathbb{C}) \rightarrow \mathrm{Aut}(\mathbb{P}^4) \cong \mathrm{PGL}_5(\mathbb{C}).$$

Then there is a finite subgroup $G \subset \mathrm{SL}_5(\mathbb{C})$ such that $\phi(G) = \overline{G}$. Suppose that G is primitive. Then we may assume that $Z(G) \subseteq [G, G]$ (see [\[5; 14\]](#)).

Example 5.1 (cf [Appendix A](#)) Let \mathbb{H} be the Heisenberg group of all unipotent (3×3) -matrices with entries in \mathbb{F}_5 . Then there is a monomorphism $\rho: \mathbb{H} \rightarrow \mathrm{SL}_5(\mathbb{C})$. Let \mathbb{HM} be the normalizer of the subgroup $\rho(\mathbb{H}) \subset \mathrm{SL}_5(\mathbb{C})$. Then there is an exact sequence

$$1 \longrightarrow \mathbb{H} \xrightarrow{\alpha} \mathbb{HM} \xrightarrow{\beta} \mathrm{SL}_2(\mathbb{F}_5) \longrightarrow 1,$$

and \mathbb{HM} is a primitive subgroup in $\mathrm{SL}_5(\mathbb{C})$ (see [\[5, Theorem 9A; 17\]](#)).

Theorem 5.2 (See [5; 14, Section 8.5].) *Let G be a finite primitive subgroup in $\mathrm{SL}_5(\mathbb{C})$ such that $Z(G) \subseteq [G, G]$. Then G is one of the groups A_5 , A_6 , S_5 , S_6 , $\mathrm{PSL}_2(\mathbb{F}_{11})$, $\mathrm{PSp}_4(\mathbb{F}_3)$, or, in the notation of [Example 5.1](#), a primitive subgroup of $\mathbb{H}\mathbb{M}$ that contains $\alpha(\mathbb{H})$.*

Note that if there are two monomorphisms $\iota_1: G \rightarrow \mathrm{SL}_5(\mathbb{C})$ and $\iota_2: G \rightarrow \mathrm{SL}_5(\mathbb{C})$ such that both subgroups $\iota_1(G)$ and $\iota_2(G)$ are primitive, then $\iota_1(G)$ and $\iota_2(G)$ are conjugate.

Lemma 5.3 *Suppose that G is one of the following groups: A_5 , A_6 , S_5 , S_6 , $\mathrm{PSL}_2(\mathbb{F}_{11})$ or $\mathrm{PSp}_4(\mathbb{F}_3)$. Then G has an invariant of degree at most 4, which implies that $\mathrm{lct}(\mathbb{P}^4, \bar{G}) \leq 4/5$.*

Proof If G is A_5 , A_6 , S_5 or S_6 , then G has an invariant of degree 2 by [Remark 2.12](#). If $G \cong \mathrm{PSp}_4(\mathbb{F}_3)$, then G has an invariant of degree 4 (see [7]). If $G \cong \mathrm{PSL}_2(\mathbb{F}_{11})$, then G has an invariant of degree 3 (see [1]). \square

Lemma 5.4 *In the notation of [Example 5.1](#), suppose that $\alpha(\mathbb{H}) \subsetneq G \subseteq \mathbb{H}\mathbb{M}$. Then G has no semi-invariants of degree at most 5 if and only if either $G = \mathbb{H}\mathbb{M}$ or G is a subgroup of $\mathbb{H}\mathbb{M}$ of index 5.*

Proof Let V be the vector space of \mathbb{H} -invariant forms of degree 5. Then the group $\mathbb{H}\mathbb{M}/\alpha(\mathbb{H}) \cong \mathrm{SL}_2(\mathbb{F}_5) \cong 2.A_5$ naturally acts on the vector space V . Moreover, it follows from [17, Theorem 3.5] that $V = V' \oplus V''$, where V' and V'' are three-dimensional $\mathrm{im}(\beta)$ -invariant linear subspaces that arise from two nonequivalent three-dimensional representations of the group A_5 , respectively. Therefore, we see that G has a semi-invariant of degree 5 if and only if V' has a $\beta(G)$ -invariant one-dimensional subspace.

Let $Z \cong \mathbb{Z}_2$ be the center of the group $\mathbb{H}\mathbb{M}/\alpha(\mathbb{H}) \cong 2.A_5$. Then $2.A_5/Z \cong A_5$. Moreover, either $\beta(G)$ is cyclic, or $Z \subseteq \beta(G)$ and $\beta(G)/Z$ is one of the following subgroups of A_5 : dihedral group of order 6, dihedral group of order 10, the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, the group A_4 , the group A_5 .

If $\beta(G)$ is cyclic, then V' is a sum of one-dimensional $\beta(G)$ -invariant linear subspaces. Hence we may assume that $Z \subseteq \beta(G)$. Recall that $Z \cong \mathbb{Z}_2$ acts trivially on V' . Thus, if $\beta(G)/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then V' is a sum of one-dimensional $\beta(G)$ -invariant subspaces.

If $\beta(G)/Z$ is a dihedral group, then V' must have one-dimensional $\beta(G)$ -invariant subspace, because irreducible representations of dihedral groups are one-dimensional or two-dimensional.

If $\beta(G)/Z \cong A_5$ or $\beta(G)/Z \cong A_4$, then V' is an irreducible representation of $\beta(G)/Z$, which implies that V' is an irreducible representation of the group $\beta(G)$. Now using [Corollary A.2](#), we complete the proof. \square

The main purpose of this section is to prove the following result.

Theorem 5.5 *In the notation of [Example 5.1](#), let G be a subgroup of the group \mathbb{HM} of index 5. Then $\text{lct}(\mathbb{P}^4, \bar{G}) \geq 6/5$.*

Combining the previous results we obtain the following.

Theorem 5.6 *Let G be a finite subgroup in $\text{SL}_5(\mathbb{C})$. Then the following conditions are equivalent:*

- *The singularity $(V \ni O)$ is exceptional.*
- *The inequality $\text{lct}(\mathbb{P}^4, \bar{G}) \geq 6/5$ holds.*
- *The group G is primitive and G does not have semi-invariants of degree at most 5.*
- *In the notation of [Example 5.1](#), either $G \cong \mathbb{HM}$ or G is isomorphic to a subgroup of the group \mathbb{HM} of index 5.*

Proof The required assertion follows from Theorems [1.17](#), [5.5](#), [5.2](#) and Lemmas [5.4](#) and [5.3](#). \square

In the remaining part of this section we will prove [Theorem 5.5](#). Let us use the notation of [Example 5.1](#). Suppose that G be a subgroup of the group \mathbb{HM} of index 5.

Lemma 5.7 *Let Λ be a \bar{G} -invariant subset of \mathbb{P}^4 . Then Λ consists of at least 10 points.*

Proof The required assertion follows from [Lemma 5.4](#) and [Corollary A.2](#). \square

Suppose that $\text{lct}(\mathbb{P}^4, \bar{G}) < 6/5$. Let us derive a contradiction.

There is a rational positive number $\lambda < 6/5$ and an effective \bar{G} -invariant \mathbb{Q} -divisor D on \mathbb{P}^5 such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^4}$ and the log pair $(\mathbb{P}^4, \lambda D)$ is strictly log canonical. Let S be an irreducible subvariety of \mathbb{P}^4 that is a minimal center in $\text{LCS}(\mathbb{P}^4, \lambda D)$. Then S is either a curve or a surface by [Lemma 2.11](#).

Let Z be the \bar{G} -orbit of the subvariety $S \subset \mathbb{P}^4$, and let r be the number of irreducible components of the subvariety Z . We may assume that

$$\mathrm{LCS}(\mathbb{P}^4, \lambda D) = \bigcup_{g \in \bar{G}} \{g(S)\}$$

by [Lemma 2.8](#). Then $\mathrm{Supp}(Z) = \mathrm{LCS}(\mathbb{P}^4, \lambda D)$. It follows from [Lemma 2.6](#) that

$$S \cap g(S) \neq \emptyset \iff S = g(S)$$

for every $g \in \bar{G}$. Then $\deg(Z) = r \deg(S)$.

Let \mathcal{I} be the multiplier ideal sheaf of the log pair $(\mathbb{P}^4, \lambda D)$, and let \mathcal{L} be the log canonical singularities subscheme of the log pair $(\mathbb{P}^4, \lambda D)$. By [Theorem 2.3](#), there is an exact sequence

$$(5.8) \quad 0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(n) \otimes \mathcal{I}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(n)) \longrightarrow H^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^4}(n)) \longrightarrow 0$$

for every $n \geq 1$. Note that $Z = \mathcal{L}$ by [Remark 2.2](#).

Lemma 5.9 *The center S is not a curve.*

Proof Suppose that S is a curve. Then it follows from [Theorem 2.7](#) that S is a smooth curve of genus g such that $2g - 2 < \deg(S)$. Moreover, it follows from [Lemma 2.9](#) that $\deg(Z) \leq 10$. Then $2g - 2 < \deg(S) \leq 10$, which implies that $g \leq 5$. The curve Z is not contained in a hyperplane, because G is transitive. Then $10 \geq \deg(Z) = r \deg(S)$, which implies that $r \leq 10$.

Using [\(5.8\)](#) and the Riemann–Roch theorem, we see that

$$(5.10) \quad 5 = h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) = r(\deg(S) - g + 1),$$

because $\mathcal{L} = Z$ and $2g - 2 < \deg(S)$. Thus, either $r = 1$ or $r = 5$.

If $r = 5$, then $\deg(S) = 2$ and $g = 0$, which contradicts [\(5.10\)](#). We see that $r = 1$. Thus S is a \bar{G} -invariant irreducible curve of genus $g \leq 5$, which is impossible by [Lemma A.8](#). \square

We see that S is a surface. Then $\deg(Z) \leq 10$ by [Lemma 2.9](#). It follows from [Theorem 2.7](#) that S is normal and has at most rational singularities, and there is an effective \mathbb{Q} -divisor B_S and an ample \mathbb{Q} -divisor Δ on the surface S such that

$$K_S + B_S + \Delta \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^4}(1)|_S,$$

and the log pair (S, B_S) has Kawamata log terminal singularities. Therefore, the equality $r = 1$ holds, since two irreducible surfaces in \mathbb{P}^4 have nonempty intersection. Thus, we see that the surface $S = Z$ is \bar{G} -invariant.

Lemma 5.11 *The surface S is not contained in a hyperplane in \mathbb{P}^4 .*

Proof The required assertion follows from the fact that G is transitive. \square

Lemma 5.12 *The surface S is not contained in a quadric hypersurface in \mathbb{P}^4 .*

Proof Suppose that there is a quadric hypersurface $Q \subset \mathbb{P}^4$ such that $S \subset Q$. Then Q is irreducible by Lemma 5.11. Moreover, it follows from Lemma 5.4 that there is a quadric hypersurface $Q' \subset \mathbb{P}^4$ such that $S \subseteq Q \cap Q'$, because otherwise the quadric Q would be \bar{G} -invariant. Then Q' is irreducible by Lemma 5.11.

Suppose that $S = Q \cap Q'$. If S is nonsingular, consider a pencil \mathcal{P} generated by the quadrics Q and Q' . Then \mathcal{P} contains exactly 5 singular quadrics, which are simple quadric cones. This means that there is a \bar{G} -orbit in \mathbb{P}^4 consisting of at most 5 points, which is impossible, because G has no semi-invariants of degree up to 5. Therefore, the surface S is singular.

It follows from [16] that $|\text{Sing}(S)| \leq 4$, because S has canonical singularities since S is a complete intersection that has Kawamata log terminal singularities. But $\text{Sing}(S)$ is \bar{G} -invariant, which contradicts Lemma 5.7.

We see that $S \neq Q \cap Q'$. Therefore, it follows from Lemma 5.11 that either S is a cone over a smooth rational cubic curve, or S is a smooth cubic scroll.

If S is a cone, then its vertex is \bar{G} -invariant, which is impossible since G is transitive. Thus, we see that S is a smooth cubic scroll. Then there is a unique line $L \subset S$ such that $L^2 = -1$, which implies that L must be \bar{G} -invariant, which is again impossible, because G is transitive. \square

Let H be a hyperplane section of the surface $S \subset \mathbb{P}^4$.

Lemma 5.13 *The equalities $H \cdot H = -H \cdot K_S = 5$ and $\chi(\mathcal{O}_S) = 0$ hold.*

Proof It follows from Corollary A.2 that there is $m \geq 0$ such that $h^0(\mathcal{O}_{\mathbb{P}^4}(3) \otimes \mathcal{I}) = 5m$. Let us show that this is possible only if $H \cdot H = -H \cdot K_S = 5$ and $\chi(\mathcal{O}_S) = 0$.

It follows from the Riemann–Roch theorem and Theorem 2.3 that

$$(5.14) \quad h^0(\mathcal{O}_S(nH)) = \chi(\mathcal{O}_S(nH)) = \chi(\mathcal{O}_S) + \frac{n^2}{2}(H \cdot H) - \frac{n}{2}(H \cdot K_S)$$

for any $n \geq 1$. It follows from Lemma 5.11, the equality (5.14) and the exact sequence (5.8) that

$$(5.15) \quad 5 = h^0(\mathcal{O}_S(H)) = \chi(\mathcal{O}_S) + \frac{1}{2}(H \cdot H) - \frac{1}{2}(H \cdot K_S),$$

and it follows from [Lemma 5.12](#), the equality [\(5.14\)](#) and the exact sequence [\(5.8\)](#) that

$$(5.16) \quad 15 = h^0(\mathcal{O}_S(2H)) = \chi(\mathcal{O}_S) + 2(H \cdot H) - (H \cdot K_S).$$

It follows from [Lemmas 2.9, 5.11](#) and [5.12](#) that $4 \leq H \cdot H = \deg(S) \leq 10$.

Suppose that $H \cdot H = 10$. It follows from the equalities [\(5.15\)](#) and [\(5.16\)](#) that $\chi(\mathcal{O}_S) = 5$ and $H \cdot K_S = H \cdot H = 10$, which is impossible, because $H \sim_{\mathbb{Q}} K_S + B_S + \Delta$, where Δ is ample and B_S is effective. Thus $H \cdot H \leq 9$.

It follows from the equalities [\(5.15\)](#) and [\(5.16\)](#) that

$$H \cdot K_S = 3\chi(\mathcal{O}_S) - 5 = 3(H \cdot H) - 20.$$

It follows from the equality [\(5.14\)](#) and the exact sequence [\(5.8\)](#) that

$$h^0(\mathcal{O}_{\mathbb{P}^4}(3) \otimes \mathcal{I}) = 35 - h^0(\mathcal{O}_S(3H)) = 35 - (\chi(\mathcal{O}_S) + \frac{9}{2}(H \cdot H) - \frac{3}{2}(H \cdot K_S)) = 5m,$$

which implies that $H \cdot H = 5$, $\chi(\mathcal{O}_S) = 0$ and $H \cdot K_S = -5$, because $4 \leq H \cdot H \leq 9$. \square

Let $\pi: U \rightarrow S$ be the minimal resolution of the surface S . Then $\kappa(U) = -\infty$ and

$$1 - h^1(\mathcal{O}_U) = 1 - h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = h^2(\mathcal{O}_U) = h^0(\mathcal{O}_U(K_U)) = 0,$$

because S has rational singularities and $\kappa(U) = -\infty$ since $H \cdot K_S = -5 < 0$.

Corollary 5.17 *The surface S is birational to $E \times \mathbb{P}^1$, where E is smooth elliptic curve.*

By [Remark 2.13](#), there is a monomorphism $\xi: \bar{G} \rightarrow \text{Aut}(Y)$, which contradicts [Corollary A.11](#).

The obtained contradiction completes the proof of [Theorem 5.5](#).

Appendix A Horrocks–Mumford group

Let \mathbb{H} be the Heisenberg group of all unipotent (3×3) -matrices with entries in \mathbb{F}_5 . Then

$$\mathbb{H} = \langle x, y, z \mid x^5 = y^5 = z^5 = 1, \ xz = zx, \ yz = zy, \ xy = zyx \rangle$$

for some $x, y, z \in \mathbb{H}$. There is a monomorphism $\rho: \mathbb{H} \rightarrow \mathrm{SL}_5(\mathbb{C})$ such that

$$\rho(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \zeta & 0 & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 & 0 \\ 0 & 0 & \zeta^3 & 0 & 0 \\ 0 & 0 & 0 & \zeta^4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where ζ is a nontrivial fifth root of unity. Let us identify \mathbb{H} with $\mathrm{im}(\rho)$. Then $Z(\mathbb{H}) \cong \mathbb{Z}_5$ and

$$\begin{pmatrix} \zeta & 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & \zeta \end{pmatrix} \in Z(\mathbb{H}),$$

where $Z(\mathbb{H})$ is the center of \mathbb{H} . Let $\phi: \mathrm{GL}_5(\mathbb{C}) \rightarrow \mathrm{PGL}_5(\mathbb{C})$ be the natural projection.

Lemma A.1 [17, Section 1] *Let $\chi: \mathbb{H} \rightarrow \mathrm{GL}_N(\mathbb{C})$ be an irreducible representation of \mathbb{H} . Then either $N = 1$ and $Z(\mathbb{H}) \subseteq \ker(\chi)$, or N is divisible by 5.*

Take $n \in \mathbb{Z}_{\geq 0}$. Then \mathbb{H} naturally acts on $H^0(\mathcal{O}_{\mathbb{P}^4}(n))$.

Corollary A.2 *Let V be a \mathbb{H} -invariant subspace in $H^0(\mathcal{O}_{\mathbb{P}^4}(n))$. Then either $\dim(V)$ is divisible by 5, or n is divisible by 5.*

Let $\mathbb{HM} \subset \mathrm{SL}_5(\mathbb{C})$ be the normalizer of the subgroup \mathbb{H} . Then there is an exact sequence

$$1 \longrightarrow \mathbb{H} \xrightarrow{\alpha} \mathbb{HM} \xrightarrow{\beta} \mathrm{SL}_2(\mathbb{F}_5) \longrightarrow 1,$$

and it follows from [17, Section 1] that there is a subgroup $\mathbb{M} \subset \mathbb{HM}$ such that $\mathbb{HM} = \mathbb{H} \rtimes \mathbb{M}$ and $\mathbb{M} \cong \beta(\mathbb{M}) = \mathrm{SL}_2(\mathbb{F}_5) \cong 2.A_5$. Put $\bar{\mathbb{H}} = \phi(\mathbb{H})$ and $\overline{\mathbb{HM}} = \phi(\mathbb{HM})$. Then $\overline{\mathbb{HM}}/\bar{\mathbb{H}} \cong \mathrm{SL}_2(\mathbb{F}_5)$ and $\bar{\mathbb{H}} \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. Let $Z(\mathbb{HM})$ be the center of the group \mathbb{HM} . Then $Z(\mathbb{HM}) = Z(\mathbb{H}) \cong \mathbb{Z}_5$.

Corollary A.3 *The group $\overline{\mathbb{HM}}$ is isomorphic to $\mathbb{HM}/Z(\mathbb{HM})$.*

Let G be a subgroup of the group \mathbb{HM} of index 5. Then $G \cong \mathbb{H} \rtimes 2.A_4 \subset \mathbb{H} \rtimes 2.A_5$ and $|\bar{G}| = 600$, where $\bar{G} = \phi(G)$. Let $Z(G)$ be the center of the group G . Then $Z(G) = Z(\mathbb{HM}) = Z(\mathbb{H}) \cong \mathbb{Z}_5$.

Lemma A.4 *Let g be an element of the group \bar{G} such that $gh = hg \in \bar{G}$ for every element $h \in \bar{\mathbb{H}}$. Then $g \in \bar{\mathbb{H}}$.*

Proof The required assertion follows from [17, Section 1]. \square

Lemma A.5 *Let F be a proper normal subgroup of $2.A_4$. Then either $F \cong \mathbb{Z}_2$ is a center of the group $2.A_4$, or $F \cong \mathbb{Q}_8$, where \mathbb{Q}_8 is the quaternion group of order 8.*

Proof The only nontrivial normal subgroup of the group A_4 is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. \square

Lemma A.6 *The group $\bar{\mathbb{H}}$ contains no proper nontrivial subgroups that are normal in \bar{G} .*

Proof Let $\theta: \overline{\mathbb{H}\mathbb{M}} \rightarrow \text{Aut}(\bar{\mathbb{H}})$ be the homomorphism such that

$$\theta(g)(h) = ghg^{-1} \in \bar{\mathbb{H}}$$

for all $g \in \overline{\mathbb{H}\mathbb{M}}$ and $h \in \bar{\mathbb{H}}$. Then $\ker(\theta) = \bar{\mathbb{H}}$ by Lemma A.4.

The homomorphism θ induces an isomorphism $\tau: \mathbb{M} \rightarrow \text{SL}_2(\mathbb{F}_5)$.

Let $F \subset \mathbb{M}$ be a subgroup such that $\beta(F) = \beta(G) \cong 2.A_4$. Then $G = \mathbb{H} \rtimes F$.

Suppose that the group $\bar{\mathbb{H}}$ contains a proper nontrivial subgroup that is a normal subgroup of the group \bar{G} . Let us consider $\bar{\mathbb{H}}$ as a two-dimensional vector space over \mathbb{F}_5 . Then $\mathbb{F}_5^2 \cong \bar{\mathbb{H}} = V_0 \oplus V_1$, where V_0 and V_1 are one-dimensional $\tau(F)$ -invariant subspaces, since $|2.A_4| = 24$ is coprime to 5.

By Lemma A.4, the homomorphism τ induces a monomorphism

$$F \longrightarrow \text{GL}_1(\mathbb{F}_5) \times \text{GL}_1(\mathbb{F}_5) \cong \mathbb{Z}_4 \times \mathbb{Z}_4,$$

which implies that F is an abelian group, which is not the case. \square

Lemma A.7 *The group \bar{G} does not contain proper normal subgroups not containing $\bar{\mathbb{H}}$.*

Proof Suppose that \bar{G} contains a normal subgroup \bar{G}' such that $\bar{\mathbb{H}} \not\subseteq \bar{G}'$. Then the intersection $\bar{G}' \cap \bar{\mathbb{H}}$ consists of the identity element in G by Lemma A.6. Hence

$$\bar{G}' \cong \beta(\bar{G}') \subseteq \beta(\bar{G}) \cong 2.A_4,$$

which implies that \bar{G}' is isomorphic to a normal subgroup of the group $2.A_4$.

Let \bar{Z} be the center of \bar{G}' . Then \bar{Z} is a normal subgroup of the group \bar{G} . Thus, we have $\bar{Z} \cong \mathbb{Z}_2$ by Lemma A.5. Hence \bar{Z} is contained in the center of \bar{G} , which contradicts Lemma A.4. \square

Lemma A.8 *Let E be a smooth irreducible curve of genus $g \leq 8$. Then there is no monomorphism $\bar{G} \rightarrow \text{Aut}(E)$.*

Proof By classification of finite subgroups in $\text{PGL}_2(\mathbb{C})$ the case $g = 0$ is impossible. The cases $2 \leq g \leq 8$ are impossible by [Theorem 2.14](#). Therefore, we may assume that E is an elliptic curve.

Let us consider E as an abelian group. Then there is an exact sequence

$$1 \longrightarrow E \xrightarrow{\iota} \text{Aut}(E) \xrightarrow{\nu} \mathbb{Z}_n \longrightarrow 1$$

for some $n \in \{2, 4, 6\}$.

Suppose that there is a monomorphism $\theta: \bar{G} \rightarrow \text{Aut}(E)$. Then $\theta(\bar{\mathbb{H}}) \subset \iota(E)$, because $\iota(E)$ contains all the elements of $\text{Aut}(E)$ of order 5.

Let g be any element of \bar{G} such that $\theta(g) \in \iota(E)$. Then $\theta(g)\theta(h) = \theta(h)\theta(g)$ for every $h \in \bar{\mathbb{H}}$, because $\iota(E)$ is an abelian group, and thus $g \in \bar{\mathbb{H}}$ by [Lemma A.4](#). Hence $\theta(\bar{G}) \cap \iota(E) = \theta(\bar{\mathbb{H}})$, which implies that $\nu(\bar{G}) \cong \beta(\bar{G}) \cong 2A_4$, which is absurd. \square

The main purpose of this section is to prove the following result.

Theorem A.9 *Let E be a smooth elliptic curve. Then there is no exact sequence of groups*

$$(A.10) \quad 1 \longrightarrow G' \xrightarrow{\iota} \bar{G} \xrightarrow{\nu} G'' \longrightarrow 1,$$

where G' and G'' are subgroups of the groups $\text{Aut}(\mathbb{P}^1)$ and $\text{Aut}(E)$, respectively.

Proof Suppose that the exact sequence of groups (A.10) does exist. Then ι is not an isomorphism, because the group $\text{Aut}(\mathbb{P}^1)$ does not contain subgroups isomorphic to \bar{G} . The monomorphism ν is not an isomorphism by [Lemma A.8](#). Then $\bar{\mathbb{H}} \subset \iota(G')$ by [Lemma A.7](#). But $\text{Aut}(\mathbb{P}^1)$ contains no subgroups isomorphic to $\bar{\mathbb{H}}$, which is a contradiction. \square

Corollary A.11 *There is no monomorphism $\bar{G} \rightarrow \text{Bir}(E \times \mathbb{P}^1)$, where E is a smooth elliptic curve.*

We believe that there is a simpler proof of [Theorem A.9](#).

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